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The method of covariant symbols in curved space-time

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Abstract. Diagonal matrix elements of pseudodifferential operators are needed in order to compute effective Lagrangians and currents. For this purpose the method of symbols is often used, which however lacks manifest covariance. In this work the method of covariant symbols, introduced by Pletnev and Banin, is extended to curved space-time with arbitrary gauge and coordinate connections. For the Riemannian connection we compute the covariant symbols corresponding to external fields, the covariant derivative and the Laplacian, to fourth order in a covariant derivative expansion. This allows one to obtain the covariant symbol of general operators to the same order. The procedure is illustrated by computing the diagonal matrix element of a nontrivial operator to second order. Applications of the method are discussed.

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1 Introduction

As is known, the functional integral formulation of quantum field theory depends on the computation of the partition functional. To one loop this amounts to adding the quadratic fluctuations above a classical solution. Typically

$$Z_{\text{1-loop}} = \int D\psi(x)\, \mathrm{e}^{-\langle\psi|\hat{K}|\psi\rangle}\,, \tag{1}$$

where the quantum fluctuations are controlled by the differential operator $\hat{K}$. This operator may depend on all kinds of external fields, and typically it will contain the covariant derivative ∇_μ (with all kinds of connections) as well as other background fields, $M(x)$, i.e. $\hat{K} = K(\nabla, M)$. Formally, the Gaussian integral gives the functional determinant $\mathrm{Det}(\hat{K})$ raised to some power, which depends on the type of fields (real or complex, bosonic or fermionic). Thus, for the effective action,

$$Z = \mathrm{e}^{-\Gamma}\,, \tag{2}$$

one formally obtains

$$\Gamma_{\text{1-loop}} = c\,\mathrm{Tr}\log(\hat{K}) = c\int \mathrm{d}^d x\sqrt{g}\,\mathrm{tr}\left\langle x|\log(\hat{K})|x\right\rangle. \tag{3}$$

This brings in a pseudodifferential operator, namely, $\log(\hat{K})$, and its kernel at coincident points. Unfortunately, the logarithm does not define an ultraviolet convergent (or even one-valued) operator for any physical space-time dimension, correspondingly the kernel of $\log(\hat{K})$ diverges at coincident points, as also does its trace. If the ζ-function regularization is used [1–4], this introduces a new pseudodifferential operator, the complex power of the fluctuation operator, $(\hat{K})^s$ [5]. Its kernel $\langle x|(\hat{K})^s|y\rangle$ is an analytic entire function with respect to s provided the points x and y are different. The diagonal matrix elements $\langle x|(\hat{K})^s|x\rangle$ are meromorphic functions of s, with a finite number of poles, which depend on the order of $\hat{K}$ and the space-time dimension, but they are analytic at $s = 0$. The computation of other observables introduces further pseudodifferential operators $\hat{f} = f(\nabla, M)$ and their diagonal matrix elements. For instance, for a gauge current

$$\delta\Gamma = \int \mathrm{d}^d x\sqrt{g}\,\mathrm{tr}(J^\mu(x)\delta A_\mu(x))\,, \tag{4}$$

with fluctuation operator of the Klein–Gordon type, $\hat{K} = -\nabla_\mu\nabla^\mu + M$, at one loop one formally obtains

$$J^\mu_{\text{1-loop}}(x) = -c\left\langle x|\left\{\nabla^\mu, (\hat{K})^{-1}\right\}|x\right\rangle, \tag{5}$$

and again some regularization procedure has to be used to render the expression meaningful.

The main purpose of this work is of practical and methodological character, namely, to address the computation of diagonal matrix elements of operators of the type $\hat{f} = f(\nabla, M)$. As we have just shown such problem is ubiquitous in one-loop calculations in quantum field theory. A more concrete goal is to extend methods existing for flat space-time to curved space-time, the covariant derivative carrying gauge and coordinate connections.

A useful technique when working with pseudodifferential operators is the method of symbols [4–7]. For an operator $\hat{f}$ constructed with x^μ and ∂_μ the symbol is essentially

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the function $f(x,p)$ such that $\hat{f} =: f(x,\partial):$, where the normal order stands for writing ∂_μ at the right of x^μ. Obviously the symbol is closely related to the Wigner representation of operators [8] which is the basis of the phase space approach to quantum mechanics [9], except that Weyl normal order [10] is used instead, so that $:f(x,\partial):$ is hermitian. As will be shown below, the symbol allows one to carry out manipulations, typically expansions of various types, and directly or indirectly it has been used extensively in the computation of the one-loop effective action and related quantities, such as the heat kernel [11–16], both in flat space-time [7, 17–19] and in curved space-time [20–31]. The mathematical aspects of the symbol in Riemannian manifolds have been considered e.g. in [32]. The extension of the method of symbols to finite temperature field theory (in the imaginary time formalism where space is compactified to a circle) has been carried out in [33–36]. A further branch of mathematical physics where nowadays the symbol and the Moyal product [37] (which provides the symbol of the product of two operators) have proven useful is in non-commutative quantum field theory [38–40], where they play an important role in the construction of non-commutative versions of existing theories.

In general, the symbol is not a covariant quantity (either in the gauge or coordinate senses) since the prescription $\partial_\mu \to p_\mu$ is not covariant. Obviously, a naive covariant prescription of the type $\nabla_\mu \to p_\mu$ would not define a faithful representation of the pseudodifferential operator because, unlike p_μ, ∇_μ is a non-Abelian quantity. When the symbol is used to compute a covariant quantity, such as the effective action, covariance is recovered at the end of the computation, but is not fully manifest in intermediate steps. Actually the situation is not as bad as it seems (examples are given below) and it is usually not necessary to go to the point of splitting ∇_μ into ∂_μ plus connections (thus spoiling the geometrical meaning of the covariant derivative, a quite bold step to take even in the simplest computations, particularly in the coordinate sector), but nevertheless, it is a more or less severe nuisance. A possible way out is to use "covariant gauges", namely, Fock–Schwinger [11, 41] in the gauge sector and Riemann normal coordinates [42–44] in the coordinate sector [26]. In [45] Pletnev and Banin proposed a new method for the gauge sector in flat space-time which implements the previous gauge fixing in a convenient way. In the present work we name their construction *covariant symbol* of the pseudodifferential operator. Whereas the ordinary symbol is a function of x^μ and p_μ, the covariant symbol is actually an operator, but multiplicative in x-space, and hence equivalent to a function of x^μ. Unlike the ordinary symbol, however, the covariant symbol is non-multiplicative with respect to p_μ, that is, it contains $\partial/\partial p_\mu$. Eliminating ∇_μ at the price of introducing $\partial/\partial p_\mu$ is in principle a net gain, though, since one is trading a non-Abelian quantity for an Abelian one. Besides being manifestly gauge covariant, the covariant symbol has the interesting property that it is a representation (in the technical sense of algebra homomorphism) of the original operator. E.g., if $\overline{f}$ is the covariant symbol $\hat{f}$, $\log(\overline{f})$ is the covariant symbol of $\log(\hat{f})$. No Moyal product is required to compute the covariant symbol of the product of two operators. Unfortunately there are also drawbacks: in general, the covariant symbol cannot be computed in closed form even for differential operators and thus expansions of some type are usually required. In practice, this is not a serious disadvantage since even the ordinary symbol of non-differential operators is not obtainable in closed form. In addition to manifest covariance, the great virtue of the covariant symbol is that, due to their homomorphism property, one needs to work them out for the building blocks only, that is, the external fields M and the covariant derivative ∇_μ, and this can be done once and for all. At present, the method of covariant symbols has been successfully used in several problems [45–49].

In this work we take a step forward and extend the method of covariant symbols to the case of curved space-time.[1] The main issue now is to retain manifest coordinate covariance, in addition to gauge covariance. In fact, we find that working with the full covariant derivative as a whole (i.e., all connections included), as advocated for instance in [16], is the cleanest way to proceed both conceptually and computationally. To some extent the construction carried out in the flat space-time case can be adapted to the curved case. However, as is known, there are important technical differences between gauge and coordinate cases. All differences stem from the fact that the covariant derivative always adds a new coordinate index, and thus a quantity X and its covariant derivative $X_\mu = \nabla_\mu X$ fall in different representations of the group of diffeomorphisms. This implies, for instance, that ∇_ν acting on X_μ will contain a further term $\Gamma^\lambda_{\nu\mu}$ not present in its action on X. This is not so in the purely gauge case. When these facts are properly taken into account, and with the help of Riemann normal coordinates in an intermediate step, the construction in [45] can be extended to the general case (i.e., coordinate plus gauge symmetries). The construction holds for completely general connections in the world sector[2] (including e.g. torsion).

As we said, the covariant symbol can seldom be obtained in closed form. A natural expansion in this context is that in the number of covariant derivatives (also known as adiabatic expansion), which permits a systematic evaluation of the covariant symbol. For an operator $\hat{f} = f(\nabla, M)$ it is sufficient to compute the covariant symbols of ∇_μ and M. We do this explicitly to second order for a generic connection, and to fourth order for the particular case of the Riemannian connection in the world sector (and arbitrary connection in the gauge sector). The Laplacian is computed to the same order in the derivative expansion. The computation provides the covariant symbols in terms of elementary operators classified by their number

[1] In their original work [45] Pletnev and Banin proposed a formula including the Riemannian connection, which, however, has not yet been used in any actual application known to us. Our own proposal is unrelated to that one.

[2] In this work we will use the label *world* interchangeably with *coordinate* or *space-time* in expressions like "world tensor", "world index", etc, to refer to properties tied to indices $\mu, \nu, \ldots$, associated to natural bases, $\partial/\partial x^\mu$, of the tangent space of the space-time manifold.

of derivatives. Such operators are just all possible local covariant operators constructed with M, the field strength tensor and their covariant derivatives. There is a number of similarities between the covariant derivative expansion and the standard heat kernel expansion. Both are expected to be asymptotic at best. They are local and blind to global properties of the space-time manifold. In both cases all possible local covariant operators are expected to appear with some universal coefficients which are rational numbers (to be determined by the computation). In the standard heat kernel expansion the operators are classified by their (mass) dimension. Because ∇_μ has dimension 1, the heat kernel expansion can be obtained a posteriori by means of a subsequent reexpansion of the covariant derivative expansion [19].

The paper is organized as follows. In Sect. 2 we consider flat space-time and revise the construction of ordinary and covariant symbols in that case, as well as their use for the computation of diagonal matrix elements. In Sect. 3 we discuss the construction of ordinary symbols in curved space-time, highlighting the subtleties introduced by the presence of curvature. In Sect. 4 we extend covariant symbols to curved space-time, discuss their properties and compute them to second order in the derivative expansion for a general gauge and world connection. Section 5 is devoted to set up a systematic computation of the covariant symbols, and they are computed to fourth order for Riemannian connection. In Sect. 6 we illustrate the ideas and techniques involved by using the covariant symbols to explicitly compute the diagonal matrix elements of a concrete operator to second order. Finally, in Sect. 7 we present our conclusions. In Appendix A we summarize some of the conventions used in the work. In Appendix B we work out the same calculation as in Sect. 6 but using the method of (ordinary) symbols. In Appendix C it is shown how to reduce momentum integrations in curved space-time to those of the flat case.

2 Symbols and covariant symbols in flat space-time

In this section we consider a d-dimensional flat space-time. The operators act on states $\psi(x)$ which can be thought of as "matter fields", as opposed to background external fields appearing in the operators. In addition to their space-time dependence, the matter fields may carry internal indices (however, for simplicity, we disregard possible world indices in the fields throughout this section). For concreteness, in what follows, we will assume that the states are vectors in the fundamental representation of some internal symmetry gauge group, and that the operators map them into the same gauge representation. The scalar product takes the form $\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^d x\, \psi_1^\dagger(x)\psi_2(x)$.

The pseudodifferential operators to be considered are of the form $\hat f = f(D, M)$. They are constructed algebraically[3] out of the covariant derivative D_μ and one or more multiplicative operators which are collectively denoted by M. Such M are just equivalent to matrix-valued (in internal space) functions of x acting as $M(x)\psi(x)$. The covariant derivative is of the form $D_\mu = \partial_\mu + A_\mu(x)$, the gauge connection $A_\mu(x)$ being also a matrix-valued function.

Under a gauge transformation $\psi(x) \to \Omega_{\rm g}^{-1}(x)\psi(x)$, where $\Omega_{\rm g}$ is a multiplicative operator and $\Omega_{\rm g}(x)$ a matrix in the internal space. Correspondingly M, D_μ and $\hat f$ transform under a similarity transformation

$$\begin{aligned} M &\to \Omega_{\rm g}^{-1} M \Omega_{\rm g}\,, \\ D_\mu &\to \Omega_{\rm g}^{-1} D_\mu \Omega_{\rm g}\,, \\ A_\mu &\to \Omega_{\rm g}^{-1}[\partial_\mu, \Omega_{\rm g}] + \Omega_{\rm g}^{-1} A_\mu \Omega_{\rm g}\,, \\ \hat f &\to \Omega_{\rm g}^{-1}\hat f\, \Omega_{\rm g}\,. \end{aligned} \tag{6}$$

(The last equality being a consequence of the fact that $\hat f$ is algebraically a function of M and D_μ.)

We can consider a basis of states of the form $|x, a\rangle$ with spatial part equal to a Dirac delta located at x and a being a gauge index, and the corresponding dual basis, $\langle x, a|y, b\rangle = \delta(x-y)\delta^a_b$. In what follows we will refer to diagonal matrix elements of an operator $\hat f$ to mean those matrix elements of the type $\langle x, a|\hat f|x, b\rangle$ (a and b not necessarily equal). For convenience, we will occasionally write the same matrix element omitting the internal indices, i.e. $\langle x|\hat f|x\rangle$, and using a matrix notation in internal space. The diagonal matrix element is gauge covariant,

$$\langle x|\hat f|x\rangle \to \Omega_{\rm g}^{-1}(x)\langle x|\hat f|x\rangle\Omega_{\rm g}(x)\,, \tag{7}$$

due to $\Omega_{\rm g}|x\rangle = \Omega_{\rm g}(x)|x\rangle$. Of course, this is somewhat formal as $\langle x|\hat f|x\rangle$ does not exist for many otherwise decent operators due to ultraviolet divergences in taking the diagonal limit. Throughout this work we will assume that the function f is sufficiently convergent so that the matrix element exists, or that a gauge invariant prescription, such as dimensional renormalization or ζ-function regularization has been used. Such a prescription always exists for symmetries as gauge invariance, which correspond to similarity transformations of $\hat f$.

2.1 Symbol of an operator

In order to compute $\langle x|\hat f|x\rangle$, a standard technique is the method of symbols [4]. Let $|0\rangle$ denote the wavefunction equal to one for all x, that is

$$\langle x, a|0, b\rangle = \delta^a_b\,, \qquad |0, a\rangle = \int \mathrm{d}^d x|x, a\rangle\,. \tag{8}$$

Then, for a given point x_0,

$$\begin{aligned} \langle x_0|f(D, M)|x_0\rangle &= \int \mathrm{d}^d y\, \delta(y - x_0)\langle x_0|f(D, M)|y\rangle \\ &= \int \mathrm{d}^d y \frac{\mathrm{d}^d p}{(2\pi)^d} \mathrm{e}^{p(y-x_0)}\langle x_0|f(D, M)|y\rangle \end{aligned}$$

[3] This means that $\hat f$ has the same algebraic properties as a sum of products of M's and D's weighted with c-number coefficients, e.g., $D_\mu \log(D^2 + M)$.

$$= \int \mathrm{d}^d y \frac{\mathrm{d}^d p}{(2\pi)^d} \langle x_0 | \mathrm{e}^{-px} f(D,M) \mathrm{e}^{px} | y \rangle$$
$$= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \langle x_0 | \mathrm{e}^{-px} f(D,M) \mathrm{e}^{px} | 0 \rangle$$
$$= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \langle x_0 | f(D+p,M) | 0 \rangle \,. \qquad (9)$$

In the second line $py := p_\mu y^\mu$. Throughout this work[4] we will use an imaginary momentum variable $p_\mu = \mathrm{i}\mathrm{p}_\mu$ (p_μ real) to save unnecessary i factors; however, $\mathrm{d}^d p := d^d \mathrm{p}$ is the standard integration in $\mathbb{R}^d$. In the third line x^μ is the position operator. In the fourth line we use the identity (8). Finally, in the last line we use the properties

$$\mathrm{e}^{-px} M(x) \mathrm{e}^{px} = M(x) \,, \quad \mathrm{e}^{-px} \partial_\mu \mathrm{e}^{px} = \partial_\mu + p_\mu \,. \qquad (10)$$

The quantity $\langle x | f(D+p,M) | 0 \rangle$ is known as the *symbol* of the pseudodifferential operator $\hat{f}$. It is a matrix-valued function of x and p. For any multiplicative operator, the property

$$\langle x_0 | h(x) | 0 \rangle = h(x_0) \qquad (11)$$

implies that the symbol of the operator M is just the function $M(x)$. More generally, because $\partial_\mu |0\rangle = 0$, the symbol of $f(D,M)$ can be obtained by dragging the ∂_μ to the right and replacing them by p_μ.

The matrix element $\langle x | f(D,M) | x \rangle$ is potentially ultraviolet divergent. Using (9) the divergence is now controlled by the momentum integration and one can make expansions or other manipulations using the symbol of the operator. On the other hand, $\langle x | f(D,M) | x \rangle$ is manifestly gauge covariant (cf. (7)) but the symbol is not, in general, due to the non-covariance of $|0\rangle$. This implies that explicit gauge covariance in (9) is only recovered after momentum integration, but not in intermediate steps. Since this is an important point let us dwell a bit on it. Clearly, if an operator $\hat{h}$ is both covariant and multiplicative, the matrix element $\langle x | \hat{h} | 0 \rangle$ will also be covariant[5]. For instance, $\langle x | D_\mu | 0 \rangle = A_\mu(x)$ (not covariant) whereas the matrix element $\langle x | \, | 0 \rangle$ of the multiplicative operator $F_{\mu\nu} := [D_\mu, D_\mu]$ is just $F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (covariant). So the lack of covariance of the symbol stems from the fact that the operator $f(D+p,M)$ (although covariant) is not multiplicative. On the other hand, let

$$\hat{f}' := \int \frac{\mathrm{d}^d p}{(2\pi)^d} f(D+p,M) \,, \qquad (12)$$

so that (9) becomes

$$\langle x_0 | \hat{f} | x_0 \rangle = \langle x_0 | \hat{f}' | 0 \rangle \,. \qquad (13)$$

The operator $\hat{f}'$ is multiplicative (in addition to covariant). This can be seen as follows. For any (imaginary) constant c-number a_μ

$$\mathrm{e}^{-ax} \hat{f}' \mathrm{e}^{ax} = \int \frac{\mathrm{d}^d p}{(2\pi)^d} f(D+p+a,M) = \hat{f}' \,, \qquad (14)$$

and this implies that $\hat{f}'$ is multiplicative. The operation in (12) projects the multiplicative component of $\hat{f}$. Another observation is that, as can be seen from (14), an operator is multiplicative if and only if it is invariant under the replacement $D_\mu \to D_\mu + a_\mu$, where a_μ is constant c-number, and in turn, this is true if and only if all D_μ appear only in the form $[D_\mu, \,]$. Some of these arguments need to be modified in the curved case (see e.g. the discussion of $Z^0_{\mu\nu}$ in (26)).

A standard technique for using the symbol in the computation of diagonal matrix elements of concrete operators (e.g. the heat kernel) is as follows: $f(D+p,M)$ is expanded in powers of D_μ and M. Each term so obtained is worked out by dragging the D_μ to the right (or to the left) producing commutators of the type $[D_\mu, \,]$, which are gauge covariant and multiplicative. At the end, there will be two type of summands, namely, the following. i) We have those where all D_μ are inside commutators. These are multiplicative and so give gauge covariant contributions to the symbol. ii) There are summands where the D_μ at the right cannot be arranged in commutators. These are non-multiplicative and break gauge covariance of the symbol. From the previous discussion it follows that such terms cancel after momentum integration and the surviving terms yield a covariant diagonal matrix element.

The method just described is illustrated in Appendix B for the more involved case of curved space-time.[6]

2.2 Covariant symbol

To achieve manifest gauge invariance prior to momentum integration in (9), one can choose to work in the covariant Fock–Schwinger gauge referred to the point x_0. An equivalent but more convenient procedure was devised by Pletnev and Banin [45] who introduced what we will call the (gauge) *covariant symbol* of an operator. This is defined as follows:

$$\overline{f} = \mathrm{e}^{-\partial_p D} \mathrm{e}^{-px} \hat{f} \mathrm{e}^{px} \mathrm{e}^{\partial_p D} \,, \qquad (15)$$

where $\partial_p D = \partial_p^\mu D_\mu$ and $\partial_p^\mu = \partial / \partial p_\mu$. (Note that $\partial_p D = D \partial_p$ just means the product of two operators; no derivative of one on the other is implied.) Therefore, while the original operator $\hat{f}$ acts on functions $\psi(x)$, its covariant symbol $\overline{f}$ is an operator on functions $\psi(x,p)$. Key properties of the covariant symbol are i) it is a multiplicative operator in x

[4] Our notational conventions are summarized in Appendix A.

[5] Indeed, $\langle x | \hat{h} | 0 \rangle \to \langle x | \Omega_{\mathrm{g}}^{-1} \hat{h} \Omega_{\mathrm{g}} | 0 \rangle = \Omega_{\mathrm{g}}^{-1}(x) \hat{h}(x) \Omega_{\mathrm{g}}(x) = \Omega_{\mathrm{g}}^{-1}(x) \langle x | \hat{h} | 0 \rangle \Omega_{\mathrm{g}}(x)$, since $\Omega_{\mathrm{g}}^{-1} \hat{h} \Omega_{\mathrm{g}}$ is also a multiplicative operator.

[6] To revert the calculation in Appendix B to the flat space-time case amounts to replace ∇_μ with D_μ, $g_{\mu\nu}$ with $\delta_{\mu\nu}$, and to set to zero all $p_{\mu\nu\cdots}$ (having two or more indices) as well as all Riemann tensors.

space, ii) it is gauge covariant and iii) it is related to the original operator by a similarity transformation.

Property iii) is obvious from its definition. ii) is also clear, since $\hat{f}$, x^μ, p_μ, ∂_p^μ, and D_μ are all gauge covariant. Property i) holds provided the original operator $\hat{f}$ does not contain p_μ and means that in the covariant symbol all ∂_μ^x appear in commutators only. The multiplicative property is equivalent to the statement $[x^\mu, \overline{f}] = 0$. That this is the case can be verified directly from the definition. Alternatively, using the property $\overline{\partial_p^\mu} = x^\mu$, which is easily verified, one has

$$\left[x^\mu, \overline{f}\,\right] = \left[\overline{\partial_p^\mu}, \overline{f}\right] = \overline{\left[\partial_p^\mu, \hat{f}\right]} = 0\,. \tag{16}$$

In the second equality we have used (iii). The multiplicative property of the covariant symbol plays an important role in what follows.

The covariant symbol can be used to compute the diagonal matrix element:

$$\begin{aligned}\left\langle x_0|\hat{f}|x_0\right\rangle &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\left\langle x_0|\mathrm{e}^{-px}\hat{f}\mathrm{e}^{px}|0\right\rangle \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\left\langle x_0|\mathrm{e}^{-\partial_p D}\mathrm{e}^{-px}\hat{f}\mathrm{e}^{px}\mathrm{e}^{\partial_p D}|0\right\rangle \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\overline{f}(x_0)\,.\end{aligned} \tag{17}$$

The second equality follows from noting that ∂_p^μ at the right vanishes (since it is acting on a p-independent wavefunction in (x,p) space). Likewise, ∂_p^μ at the left vanishes due to integration by parts.[7] In the last equality we have used that the covariant symbol is multiplicative to write $\overline{f}(x_0)$ instead of the matrix element $\langle x_0|\overline{f}|0\rangle$.

As we can see from the properties i-iii), the mapping $\hat{f} \to \overline{f}$ really defines a representation of the operators $\hat{f}$ in x-space in terms of multiplicative operators (with respect to x) in (x,p)-space, which is consistent with gauge covariance. A further property is that the covariant symbol preserves the hermiticity properties of the original operator.

Because the covariant symbol is a representation (i.e. an algebra homomorphism) one has e.g.

$$\begin{aligned}\overline{f(D,M)} &= f(\overline{D},\overline{M})\,,\\ \overline{F}_{\mu\nu} &= [\overline{D}_\mu,\overline{D}_\nu]\,,\\ \overline{[D_\mu,M]} &= [\overline{D}_\mu,\overline{M}]\,,\end{aligned} \tag{18}$$

and so on. Using the definition (15), the basic operators $\overline{D}_\mu$ and $\overline{M}$ can be readily computed in terms of a covariant derivative expansion [45]

$$\begin{aligned}\overline{M} &= \mathrm{e}^{-[\partial_p D,\,]}M\\ &= M - M_\mu\,\partial_p^\mu + \frac{1}{2!}M_{\nu\mu}\,\partial_p^\nu\partial_p^\mu - \frac{1}{3!}M_{\alpha\nu\mu}\,\partial_p^\alpha\partial_p^\nu\partial_p^\mu + \cdots\,,\\ \overline{D}_\lambda &= \mathrm{e}^{-[\partial_p D,\,]}(D_\lambda + p_\lambda)\\ &= p_\lambda - \frac{1}{2!}F_{\mu\lambda}\,\partial_p^\mu + \frac{2}{3!}F_{\nu\mu\lambda}\,\partial_p^\nu\partial_p^\mu - \frac{3}{4!}F_{\alpha\nu\mu\lambda}\,\partial_p^\alpha\partial_p^\nu\partial_p^\mu\\ &\quad + \cdots\end{aligned} \tag{19}$$

In writing these formulas we have denoted the derivatives of M and $F_{\mu\nu}$ by introducing the convention $[D_\mu, X_I] = X_{\mu I}$, i.e.,

$$[D_\mu, X_{\alpha_1\cdots\alpha_n}] = X_{\mu\alpha_1\cdots\alpha_n}\,. \tag{20}$$

As is readily verified, the expansions in (19) are consistent with the last two equations in (18). ($\overline{F}_{\mu\nu}$ follows the same formula as $\overline{M}$ since the latter only assumes M to be a multiplicative operator.)

As a further convention, we will exploit the fact that the derivatives ∂_p^μ commute and so all their indices are symmetrized, to use a single symbol s for all of them, that is, we will often write

$$\begin{aligned}\overline{M} &= M - M_s\,\partial_p^s + \frac{1}{2!}M_{ss}\,(\partial_p^s)^2 - \frac{1}{3!}M_{sss}\,(\partial_p^s)^3 + \cdots\,,\\ \overline{D}_\lambda &= p_\lambda - \frac{1}{2!}F_{s\lambda}\,\partial_p^s + \frac{2}{3!}F_{ss\lambda}\,(\partial_p^s)^2 - \frac{3}{4!}F_{sss\lambda}\,(\partial_p^s)^3 + \cdots\end{aligned} \tag{21}$$

The use of covariant symbols to compute diagonal matrix elements is illustrated in Sect. 6 for curved space-time. Since it is easy to reduce that calculation to the simpler case of flat space-time (see footnote 6) we do give further examples here.

Note that the covariant symbol method is compatible with derivative expansions (see [48, 50] for strict derivative expansions of the effective action functional of Dirac fermions using this method). Such expansions are expected to be asymptotic in general.

Another comment has to do with momentum space integration by parts. Formally, $f(D+p,M)$ and $\overline{f} = \mathrm{e}^{-\partial_p D}f(D+p,M)\mathrm{e}^{\partial_p D}$ would differ by terms with ∂_p, implying that the difference should vanish on $|0_p\rangle$ or $\langle 0_p|$ (see footnote 7). This formal argument is correct for sufficiently well behaved operators in the ultraviolet, e.g. M, but not for D_μ in (21) (see also $\overline{\nabla_\mu\nabla^{\mu(2)}}$ in (112)). Of course, it is never necessary to take diagonal matrix elements of divergent operators (without some regularization to make them convergent). The homomorphism property implies that $\overline{D}_\lambda$ provides the suitable momentum dependence to give the correct result when used as part of an ultraviolet convergent operator.

To summarize this section, the ordinary symbols are representations of pseudodifferential operators in terms of functions of x and p which are matrix-valued in internal space and they are not gauge covariant. The representation introduced by Pletnev and Banin in flat space-time, on the other hand, is in terms of operators which are multiplicative with respect to x and so equivalent to functions. In

[7] A more precise form of (17) would be

$$\left\langle x|\hat{f}|x\right\rangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\left\langle x|\left\langle p|\overline{f}|0\right\rangle|0_p\right\rangle = \left\langle x|\left\langle 0_p|\overline{f}|0\right\rangle|0_p\right\rangle\,,$$

where $|0_p\rangle$ stands for the unit wavefunction in p space. The statement is then $\mathrm{e}^{\partial_p D}|0_p\rangle = |0_p\rangle$ and $\langle 0_p|\mathrm{e}^{-\partial_p D} = \langle 0_p|$.

this sense they are similar to the ordinary symbols (which also remain operators in internal space). They are covariant and enjoy the homomorphism property, at the price of being non-multiplicative in p space. Both ordinary and covariant symbols provide diagonal matrix elements upon integration over p.

3 Symbols in curved space-time

3.1 General considerations

The method of symbols can be extended to curved space-time. The main issue now is to preserve both gauge covariance and coordinate or world[8] covariance. The space-times we consider may have Euclidean or Minkowskian signatures. We will treat the two cases simultaneously since there is no formal difference for our purposes. We will often refer to the Riemannian connection to mean the unique torsionless metric preserving connection, regardless of the signature of the metric.

The pseudodifferential operator is now of the form $\hat{f} = f(\nabla, M)$ where ∇_μ is the covariant derivative and includes connections for the parallel transport of all indices: world and internal indices. The latter include gauge, Lorentz frame, Dirac indices in the case of fermions, and so on.[9] In what follows we use indifferently an "internal" or "gauge" index to mean any kind of internal index. The matter fields $\psi(x)$ may contain internal indices as well as world indices. Likewise, the external fields M may also contain all kind of indices and act as multiplicative operators with respect to x. The metric $g_{\mu\nu}(x)$ is an example of such a field.

We do not assume that $f(\nabla, M)$ should be a world scalar (cf. Sect. 6 for an example). As a consequence $\hat{f}$ may connect different diffeomorphism representations. The reason for this generality is that there is no net gain in restricting oneself to the equal representation case. This is because we need to consider covariant symbols not only of the final operator $\hat{f}$ but also of ∇_μ as a building block and ∇_μ always connects different tensor representations. This is an important difference with the gauge case where one can work consistently viewing all operators as matrices in internal space.[10] Of course, it would be pointless to try to erase the difference between gauge and coordinate cases using a tetrad, e.g. $D_a = \mathrm{e}_a^\mu \nabla_\mu$ [52] since, although D_a is a world scalar, it connects now different internal representations (namely, with respect to the new internal structure introduced by the Lorentz index a).

For states in equal representations the scalar product is

$$\langle \psi_1 | \psi_2 \rangle = \int \mathrm{d}^d x \sqrt{g(x)}\, \psi_1^\dagger(x) \psi_2(x)\,, \tag{22}$$

$g(x)$ denoting $|\det g_{\mu\nu}|$. As usual the scalar product has been defined so that it is coordinate invariant, although metric dependent. (Note however that, as shown below, the construction of the covariant symbols themselves do not require a metric to be defined.) For states in different representations the scalar product vanishes. An active world (or coordinate, or diffeomorphism) transformation, $x^\mu \to x'^\mu(x)$, defines a corresponding operator on states $\psi \to \hat{\Omega}_w^{-1}\psi$, which takes the form

$$\psi(x) \to \left(\hat{\Omega}_w^{-1}\psi\right)(x) = \psi\left(x'(x)\right) \tag{23}$$

for a scalar,

$$\psi_\mu(x) \to \frac{\partial x'^\alpha}{\partial x^\mu} \psi_\alpha\left(x'(x)\right) \tag{24}$$

on covariant world vectors, and so on.

As in the flat case we will use a basis of the tensor product type, with states $|x_0, a, w\rangle$ located at x_0 (wavefunction $\delta(x - x_0)/\sqrt{g(x)}$), a being a gauge index and w a set of world indices (empty for world scalar states), with dual basis $\langle x, a, w | y, b, w' \rangle = \delta_b^a \delta_{w'}^w \delta(x-y)/\sqrt{g(x)}$. The metric in the space-time factor of the basis states is introduced so that they are world scalars, and similarly for the scalar product.

Once again we want to evaluate diagonal (in x) matrix elements $\langle x, a, w | \hat{f} | x, b, w' \rangle$. For short we will often write just $\langle x | \hat{f} | x \rangle$; however, one should keep in mind the presence of world and internal indices since they determine how the covariant derivative acts. In particular, the operator $\hat{f}$ should connect the in representation (b, w') with the out representation (a, w), so that $\langle \psi_1 | \hat{f} | \psi_2 \rangle$ is a gauge and world singlet.

As in the purely gauge case, we can regard the matrix element $\langle x | f(\nabla, M) | x \rangle$ as a (gauge and world) covariant function of x which takes values on operators acting on internal and world indices. And in turn this can be viewed as equivalent to a covariant multiplicative operator (in the purely gauge case, such a multiplicative operator is $\hat{f}'$ introduced in (12)).

Because multiplicative operators play an important role in what follows let us define them more precisely. A *c-number multiplicative operator*, $\hat{\phi}$, is one that acts in the form

$$\hat{\phi} | x, a, w \rangle = \phi(x) | x, a, w \rangle\,, \tag{25}$$

[8] See footnote 2.

[9] We are following the approach found for instance in [16]. In this approach, if e_μ^a is the tetrad field, the connections on the indices μ and a are such that $\nabla_\nu e_\mu^a = 0$; likewise for the Dirac gammas $\nabla_\mu \gamma^\nu = 0$, with a suitable connection acting on the Dirac indices. This convention is not universally adopted. For instance, in [51] $\nabla_\nu e_\mu^a$ would only include the connection for the world index while $D_\nu e_\mu^a$ would only include the connection for the tetrad index.

[10] In the gauge case (and flat space-time), if ψ is a gauge vector $D_\mu \psi$ is again a gauge vector and so in a second derivative $D_\nu D_\mu \psi$, D_ν would still be "the same operator" $\partial_\nu + A_\nu$. In the general covariant case (and gauge singlet), if ϕ is a world scalar, $\nabla_\mu \phi = \partial_\mu \phi$ is a world vector and ∇_ν "acts differently" on it, namely, as $\nabla_\nu = \partial_\nu - \Gamma_\nu$ (Γ_ν being a matrix on world indices $(\Gamma_\nu)_\alpha{}^\beta = \Gamma_{\nu\alpha}^\beta$). The point is, of course, that due to its geometrical meaning (through the Leibnitz rule), ∇_μ acts consistently at each place (i.e. using the proper connection), and there is no need to worry about such details.

$\phi(x)$ being a fixed complex function. Such an operator is a gauge singlet and a world scalar. Now, by definition a *multiplicative operator* is one that commutes with all c-number multiplicative operators. A multiplicative operator is diagonal in x but in general non-diagonal with respect to all other degrees of freedom. In commutators, the c-number multiplicative operators are blind to those degrees of freedom but sensible to derivatives with respect to x. Here we can see a difference between the purely gauge case and the general case (gauge plus world degrees of freedom). In the flat case, $[D_\mu, D_\nu]$ and $[D_\mu, \mathcal{Q}]$, with $\mathcal{Q}$ multiplicative, are multiplicative operators, as is easily verified. This property is lost when a world connection is included, i.e. for ∇_μ. For a generic world connection, the operator

$$Z^0_{\mu\nu} := [\nabla_\mu, \nabla_\nu]\,, \tag{26}$$

is not multiplicative, since acting on a world scalar and gauge singlet state $\phi(x)$, it gives [11]

$$[\nabla_\mu, \nabla_\nu]\phi = -T_{\mu\nu}{}^\lambda \nabla_\lambda \phi = -T_{\mu\nu}{}^\lambda \frac{\partial \phi}{\partial x^\lambda}\,, \tag{27}$$

being $T_{\mu\nu}{}^\lambda$ the torsion. (Equivalently, $[[\nabla_\mu, \nabla_\nu], \hat\phi] = -T_{\mu\nu}{}^\lambda[\nabla_\lambda, \hat\phi]$, in terms of c-number multiplicative operators.) The result depends on derivatives of ϕ and so such $Z^0_{\mu\nu}$ is not a multiplicative operator in the presence of torsion. A remedy is to introduce the new operator

$$Z_{\mu\nu} := [\nabla_\mu, \nabla_\nu] + \frac{1}{2}\left\{\nabla_\lambda, T_{\mu\nu}{}^\lambda\right\}\,, \tag{28}$$

($\{\,,\}$ denotes anticommutator) which is multiplicative, as is readily verified. It coincides with $Z^0_{\mu\nu}$ for a torsionless connection such as the Levi-Civita or Riemannian connection (in the world sector). Nevertheless, $[\nabla_\alpha, Z_{\mu\nu}]$ is again non-multiplicative (even for the Riemannian connection). In addition, at variance with $F_{\mu\nu}$ of the flat case, $Z_{\mu\nu}$ will not commute with the momentum p_μ (to be introduced subsequently, similar to the flat case). This is because $Z_{\mu\nu}$ acts on world indices

$$[Z_{\mu\nu}, p_\lambda] = -R_{\mu\nu}{}^\sigma{}_\lambda\, p_\sigma\,. \tag{29}$$

(This formula holds for any world connection.) As noted previously, the difference between D_μ and ∇_μ is due to the fact that ∇_μ acts on world indices but also adds world indices.

Because there are several types of quantities to be considered, we will introduce the following notation: the more general quantities or objects (such as operators, wavefunctions, matrix elements, etc) to be considered in this section belong to the class

$$\mathcal{C}(x, \nabla, Z, W, I, p)\,. \tag{30}$$

[11] Our conventions are such that, for a world vector gauge singlet V^λ,

$$[\nabla_\mu, \nabla_\nu]V^\lambda = +R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\lambda\,,$$
$$\mathcal{R}_{\mu\nu} := R_{\lambda\mu}{}^\lambda{}_\nu\,, \quad \mathbf{R} := \mathcal{R}^\lambda{}_\lambda\,.$$

The presence of the label x indicates that the quantity in question may depend on x^μ. Likewise, the label ∇ denotes that the object may be non-multiplicative in x space. Z means that it may contain $Z_{\mu\nu}$ or other *multiplicative operators that act on world indices*. W indicates that it may contain world indices, I that they may contain internal (or gauge, or bundle) indices. Finally, p means that it may depend on p_μ.

On the other hand, the class

$$\mathcal{C}(x, \underline{\nabla}, Z, W, \underline{I}, p)\,, \tag{31}$$

or simply $\mathcal{C}(\underline{\nabla}, \underline{I})$, will indicate quantities which are multiplicative in x space [do not contain "free" ∇_μ] (denoted $\underline{\nabla}$) and are gauge singlets [do not contain internal indices] (denoted $\underline{I}$), and similarly for other underlined labels. Thus, for instance, p_μ is in class $\mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{I})$, the operators M in $f(\nabla, M)$ are in class $\mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{p})$, and $Z_{\mu\nu}$ is in class $\mathcal{C}(\underline{\nabla}, \underline{p})$ (while $Z^0_{\mu\nu} \in \mathcal{C}(\underline{p})$ for a world connection with torsion). c-number multiplicative operators are in $\mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{I}, \underline{W})$. Multiplicative operators are in $\mathcal{C}(\underline{\nabla})$.

3.2 Diagonal matrix elements

To implement the method of symbols as for the flat case, we proceed similarly to (9), starting with the diagonal matrix element

$$\begin{aligned}\left\langle x_0|\hat f|x_0\right\rangle &= \frac{1}{\sqrt{g(x_0)}}\int \mathrm{d}^d x_1 \delta(x_1 - x_0) \\ &\quad\times \langle x_0|f(\nabla, M)|x_1\rangle \sqrt{g(x_1)} \\ &= \frac{1}{\sqrt{g(x_0)}}\int \mathrm{d}^d x_1 \frac{\mathrm{d}^d p}{(2\pi)^d}\mathrm{e}^{p(x_1 - x_0)} \\ &\quad\times \langle x_0|f(\nabla, M)|x_1\rangle \sqrt{g(x_1)}\,.\end{aligned} \tag{32}$$

The matrix element is independent of the coordinate system, but the symbol is not. So we will pick a certain *reference coordinate system* (RCS) and denote its coordinates by $\xi^A(x)$, $A = 1, \ldots, d$. These are d world scalar functions; furthermore, we set to zero the connection associated to the indices A (recall that ∇_μ is defined acting on all indices with the appropriate connection). We will reserve the symbol x^μ to denote an arbitrary coordinate system. Then

$$\nabla_\mu \xi^A = \frac{\partial \xi^A}{\partial x^\mu} =: t^A_\mu(x)\,. \tag{33}$$

These are world vector. Let us also introduce the dual contravariant world vectors

$$t^\mu_A(x) := \frac{\partial x^\mu}{\partial \xi^A}\,, \tag{34}$$

such that

$$t^A_\mu\, t^\mu_B = \delta^A_B\,, \quad t^\mu_A\, t^A_\nu = \delta^\mu_\nu\,. \tag{35}$$

Using in (32) the RCS, if ξ^A_0, ξ^A_1, and $g^{(\xi)}(x)$, denote, respectively, the coordinates of the point x_0 and x_1, and the

determinant of the metric in the RCS, we find

$$\left\langle x_0|\hat{f}|x_0\right\rangle = \frac{1}{\sqrt{g^{(\xi)}(x_0)}}\int \mathrm{d}^d\xi_1 \frac{\mathrm{d}^d p_A}{(2\pi)^d}\mathrm{e}^{p_A(\xi_1^A-\xi_0^A)} \times \langle x_0|f(\nabla,M)|x_1\rangle\sqrt{g^{(\xi)}(x_1)}\,. \tag{36}$$

The momentum integration variables p_A are d c-number and space-time constant quantities, $p_A \in \mathcal{C}(\underline{x},\underline{\nabla},\underline{Z},\underline{W},\underline{I},p)$.

It is convenient to work in an arbitrary coordinate system. To this end, let us define the world vector fields

$$p_\mu = t_\mu^A p_A\,, \qquad X^\mu = t_A^\mu \xi^A\,. \tag{37}$$

The c-number function $\Phi = p_A\xi^A$ can also be written as $p_\mu X^\mu$, and moreover

$$p_\mu = \nabla_\mu \Phi\,. \tag{38}$$

We can write

$$\left\langle x_0|\hat{f}|x_0\right\rangle = \frac{1}{\sqrt{g^{(\xi)}(x_0)}}\int \mathrm{d}^d x_1 \frac{\mathrm{d}^d p_A}{(2\pi)^d}\mathrm{e}^{p_\mu(X_1^\mu - X_0^\mu)} \times \langle x_0|f(\nabla,M)|x_1\rangle\sqrt{g(x_1)}\,. \tag{39}$$

Now, $p_\mu X^\mu$ being a c-number multiplicative operator, we can apply (25) to write

$$\left\langle x_0|\hat{f}|x_0\right\rangle = \frac{1}{\sqrt{g^{(\xi)}(x_0)}}\int \mathrm{d}^d x_1 \frac{\mathrm{d}^d p_A}{(2\pi)^d} \times \left\langle x_0|\mathrm{e}^{-pX}f(\nabla,M)\mathrm{e}^{pX}|x_1\right\rangle\sqrt{g(x_1)}\,. \tag{40}$$

Introducing now the space-time constant states $|0,a,w\rangle$, which lie in the class $\mathcal{C}(\underline{x},\underline{\nabla},\underline{Z},\underline{p})$,

$$\langle x,a,w|0,b,w'\rangle = \delta_b^a\delta_{w'}^w\,, \qquad |0,a,w\rangle = \int \mathrm{d}^d x\sqrt{g(x)}|x,a,w\rangle \tag{41}$$

yields

$$\left\langle x_0|\hat{f}|x_0\right\rangle = \frac{1}{\sqrt{g^{(\xi)}(x_0)}}\int\frac{\mathrm{d}^d p_A}{(2\pi)^d}\left\langle x_0|\mathrm{e}^{-pX}f(\nabla,M)\mathrm{e}^{pX}|0\right\rangle = \frac{1}{\sqrt{g^{(\xi)}(x_0)}}\int\frac{\mathrm{d}^d p_A}{(2\pi)^d}\left\langle x_0|f(\nabla+p,M)|0\right\rangle\,, \tag{42}$$

where we have used the identity (understood as a product of three operators)

$$\mathrm{e}^{-pX}\nabla_\mu\mathrm{e}^{pX} = \nabla_\mu + p_\mu\,. \tag{43}$$

The quantity $\langle x_0|f(\nabla+p,M)|0\rangle$ is now the symbol of $\hat{f}$ at x_0. Again it is not gauge covariant, since under a local gauge transformation, $\Omega_{\mathrm{g}}(x)$, $|0\rangle$ stops being space-time constant. For a similar reason, it is not world covariant under $\hat{\Omega}_w$, unless $|0\rangle$ is a world scalar. (However, as in the gauge case, covariance is recovered for matrix elements $\langle x_0|\,|0\rangle$ of multiplicative operators.) In addition, the symbol depends on the choice of RCS through p_μ and t_μ^A. p_μ is the vector field which happens to take constant components p_A in the RCS, and similarly X^μ has components ξ^A precisely in that coordinate system. Such RCS dependence cancels after momentum integration, as $\langle x_0|f(\nabla,M)|x_0\rangle$ is gauge and coordinate covariant. As in the flat space-time case (42) can also be written as

$$\langle x_0|\hat{f}|x_0\rangle = \langle x_0|\hat{f}'|0\rangle\,, \tag{44}$$

with

$$\hat{f}' := \frac{1}{\sqrt{g^{(\xi)}(x)}}\int\frac{\mathrm{d}^d p_A}{(2\pi)^d}f(\nabla+p,M) \tag{45}$$

($g^{(\xi)}(x)$ being a c-number multiplicative operator here). The operator $\hat{f}'$ is multiplicative, formally independent of the in and out state spaces, and gauge and world covariant. Furthermore, it is RCS independent; the RCS dependence of the momentum integral through the vector field t_μ^A in p_μ exactly cancels with the prefactor $1/\sqrt{g^{(\xi)}}$ (cf. Appendix C). As the scalar product itself, $\hat{f}'$ is metric dependent.

As in the flat case, we can work out $f(\nabla+p,M)$ by dragging ∇_μ to the right.[12] A very important difference with the flat case is that ∇_μ and p_ν no longer commute. Their commutator is just the covariant derivative of p_μ,

$$[\nabla_\mu,p_\nu] = \left[\nabla_\mu,t_\nu^A\right]p_A = t_{\mu\nu}^A p_A = t_A^\lambda\, t_{\mu\nu}^A\, p_\lambda \tag{46}$$

($t_{\mu\nu}^A$ being the covariant derivative of t_ν^A, according to our convention). The same computation in the RCS system, where p_μ equals p_A (and so $\partial_\mu^x p_\nu = 0$), gives

$$[\nabla_\mu,p_\nu] = -\Gamma^{(\xi)\lambda}_{\mu\nu}p_\lambda\,. \tag{47}$$

In an arbitrary coordinate system this becomes

$$[\nabla_\mu,p_\nu] = -P_\mu{}^\lambda{}_\nu p_\lambda\,, \tag{48}$$

where $P_\mu{}^\lambda{}_\nu$ is the world tensor with components precisely equal to $\Gamma^\lambda_{\mu\nu}$ in the RCS. Hence,

$$P_\mu{}^\lambda{}_\nu = -t_A^\lambda\, t_{\mu\nu}^A\,. \tag{49}$$

As we mentioned before, another difference with the flat case (and hence another complication) is that the construction $[\nabla_\mu,\ ]$ does not automatically produce multiplicative operators. In any case, after moving the ∇ to the right, in

[12] And the rule $\partial_\mu^x|0\rangle = 0$ still applies. However, as in the purely gauge case, it is not practical to lose manifest covariance. It is preferable to let the momentum integration to kill non-multiplicative (and so non-covariant) contributions. See Appendix B for an example.

principle one will be able to manage to form multiplicative combinations of nablas in some terms, plus terms in which non-multiplicative combinations appear at the right. The latter vanish upon momentum integration. Such integration will usually require one to put all p_μ *together* at the left (recall that $Z_{\mu\nu}$ and p_λ do not commute) except those in the form $p_\mu p^\mu$, appearing in propagators, etc. Note that $p_\mu p^\mu$ is not constant even in the RCS, because $p^\mu(x) = g^{\mu\nu}(x)p_\nu$; however, it is a c-number multiplicative operator, so it commutes with all multiplicative operators ($p_\mu p^\mu$ fails to commute with non-multiplicative operators, but those have been already been disposed of.) The key point is that when only multiplicative operators appear, the tensor $P_\mu{}^\lambda{}_\nu$ and its covariant derivatives will be needed only at the point $x = x_0$. Upon momentum integration these tensors will appear only through combinations which are independent of the choice of the RCS, e.g.

$$\nabla_\mu P_\nu{}^\alpha{}_\beta - \nabla_\nu P_\mu{}^\alpha{}_\beta - P_\mu{}^\alpha{}_\lambda P_\nu{}^\lambda{}_\beta + P_\nu{}^\alpha{}_\lambda P_\mu{}^\lambda{}_\beta = R_{\mu\nu}{}^\alpha{}_\beta \,. \tag{50}$$

(For the momentum integration with x-dependent $p_\mu p^\mu$ see Appendix C.) In practice, the natural way to proceed is to take as RCS the Riemann normal coordinates at $x = x_0$ from the beginning, since this choice provides manifestly covariant results for the tensor $P_\mu{}^\lambda{}_\nu$ and its covariant derivatives.

In Appendix B we illustrate all previous points for the operator $\hat{Q}_{\mu\nu}$ in (113) and the Riemann connection. The matrix elements of this operator are computed to second order in a derivative expansion using the (ordinary) symbols method.

4 Covariant symbols in curved space-time

As we have just sketched in the previous section, one can work with the ordinary symbols for pseudodifferential operators in curved space-time along the same lines as for the flat case, although things are, in general, more involved in the curve case and covariance is recovered only after momentum integration. In this section we introduce the covariant symbols in the presence of curvature. They are fully covariant representations in terms of operators which are multiplicative with respect to x.

We will need derivatives with respect to the momenta p_A. These are denoted as ∂^A:

$$\partial^A := \frac{\partial}{\partial p_A} \,, \quad \partial^\mu := t_A^\mu \, \partial^A \,, \tag{51}$$

The derivatives ∂^μ are contravariant world vectors which satisfy

$$[\partial^\mu, p_\nu] := \delta^\mu_\nu \tag{52}$$

and, as a consequence of (48),

$$[\nabla_\mu, \partial^\nu] = P_\mu{}^\nu{}_\lambda \partial^\lambda \,. \tag{53}$$

We should extend now our previous notation. The most general objects belong to the class

$$\mathcal{C}(x, \nabla, Z, W, I, p, \partial) \,, \tag{54}$$

where the new label ∂ indicates a possible dependence on ∂^A.[13] On the other hand the class, $\mathcal{C}(\underline{\partial})$ denotes quantities which are multiplicative with respect to p_A.

In the curved case we introduce a preliminary definition of the covariant symbol of an operator $\hat{f}$ as

$$\overline{f} := \mathrm{e}^{-\frac{1}{2}\{\nabla_\mu, \partial^\mu\}} \mathrm{e}^{-p_\alpha X^\alpha} \hat{f} \, \mathrm{e}^{p_\beta X^\beta} \mathrm{e}^{\frac{1}{2}\{\nabla_\nu, \partial^\nu\}} \,, \quad \hat{f} \in \mathcal{C}(\underline{p}, \underline{\partial}) \,. \tag{55}$$

This definition, as well as the general properties to be discussed below, holds actually for any connection on the world indices, although eventually we will restrict ourselves to the Riemannian connection for simplicity. As shown subsequently, the map $\hat{f} \mapsto \overline{f}$ defines an operator representation from $\mathcal{C}(x, \nabla, Z, W, I, \underline{p}, \underline{\partial})$ into $\mathcal{C}(x, \underline{\nabla}, Z, W, I, p, \partial)$.

The definition depends on the choice of the (arbitrary) RCS (in which the tensors p_μ and ∂^μ have constant components). Eventually we will take the RCS as the Riemann normal coordinate system, thereby obtaining fully covariant expressions for the covariant symbol. Because ∇_μ and ∂^μ do not commute (cf. (53)), one can extend the construction $\mathrm{e}^{D_\mu \partial^\mu}$ corresponding to the flat case in several different ways, among others, as $\mathrm{e}^{\nabla_\mu \partial^\mu}$ or $\mathrm{e}^{\partial^\mu \nabla_\mu}$, or even as $\mathrm{e}^{\frac{1}{2}\{\nabla_\mu, \partial^\mu\}}$. All of them are valid. The two former choices give slightly simpler formulas, but the latter has the virtue of preserving the hermiticity properties of the original operator, e^{pX} being unitary.

The use of the covariant symbol to compute the diagonal matrix element is fully analogous to its flat space-time version

$$\begin{aligned} \left\langle x_0 | \hat{f} | x_0 \right\rangle &= \frac{1}{\sqrt{g^{(\xi)}(x_0)}} \int \frac{\mathrm{d}^d p_A}{(2\pi)^d} \left\langle x_0 | \mathrm{e}^{-pX} \hat{f} \mathrm{e}^{pX} | 0 \right\rangle \\ &= \frac{1}{\sqrt{g^{(\xi)}(x_0)}} \int \frac{\mathrm{d}^d p_A}{(2\pi)^d} \\ &\quad \times \left\langle x_0 | \mathrm{e}^{-\frac{1}{2}\{\nabla, \partial\}} \mathrm{e}^{-pX} \hat{f} \mathrm{e}^{pX} \mathrm{e}^{\frac{1}{2}\{\nabla, \partial\}} | 0 \right\rangle \\ &= \frac{1}{\sqrt{g^{(\xi)}(x_0)}} \int \frac{\mathrm{d}^d p_A}{(2\pi)^d} \left\langle x_0 | \overline{f} | 0 \right\rangle \,. \end{aligned} \tag{56}$$

In the second equality we have used

$$\frac{1}{2}\{\nabla_\mu, \partial^\mu\} = \nabla_\mu \partial^\mu - \frac{1}{2} P_\mu{}^\mu{}_\lambda \partial^\lambda = \partial^\mu \nabla_\mu + \partial^\lambda \frac{1}{2} P_\mu{}^\mu{}_\lambda \,, \tag{57}$$

and so the rules $\partial^A |0_p\rangle = \langle 0_p | \partial^A = 0$ can be exploited as in the flat case ($|0_p\rangle$ being the unit wavefunction in p space; see footnote 7).

[13] From now on we use ∂^μ to denote ∂^μ_p since the non-covariant operator $\partial^x_\mu = \partial/\partial x^\mu$ will appear rarely.

Because the covariant symbol is multiplicative with respect to x (to be shown subsequently), one could, loosely speaking, replace $\langle x_0|\overline{f}|0\rangle$ with $\overline{f}(x_0)$, interpreted as an operator valued function at x_0. In addition, in the absence of derivatives, whether p_μ is constant or not is no longer relevant, and one can formally integrate over p_μ instead of p_A, the Jacobian implying the replacement of $g^{(\xi)}(x_0)$ with $g(x_0)$. We have

$$\left\langle x_0|\hat{f}|x_0\right\rangle = \frac{1}{\sqrt{g(x_0)}}\int \frac{\mathrm{d}^d p_\mu}{(2\pi)^d}\,\overline{f}(x_0)\,. \tag{58}$$

In case of ambiguity the expression (56) should be used. (See also Sect. 6 for further details.)

The properties of the covariant symbol are as follows.

i) It is a representation (an algebra homomorphism). This follows from it being defined as a similarity transformation. Actually, the definition in (55) is a similarity transformation in an extended sense, since in general the ∇_μ in the formula will fall in different representations (different rank world tensors). In any case, whenever ∇_μ acts on a field, it selects, by convention, the appropriate connections corresponding to the gauge and world representation of the field, in such a way that the homomorphism property holds, that is[14]

$$\lambda X+\mu Y \mapsto \lambda\overline{X}+\mu\overline{Y}\,,\quad XY\mapsto\overline{XY}\,. \tag{59}$$

ii) The covariant symbol is a multiplicative operator (with respect to x), i.e., it falls in the class $\mathcal{C}(\underline{\nabla})$, provided the original operator does not act in p space, or more precisely $\hat{f}\in\mathcal{C}(\underline{p})$ (it may contain ∂^μ). This can be seen as follows: the covariant symbol being multiplicative is equivalent to

$$\mathrm{e}^{-aX}\overline{f}\mathrm{e}^{aX}=\overline{f}\,,\quad \forall a_A\in\mathcal{C}(\underline{x},\underline{\nabla},\underline{Z},\underline{W},\underline{I},\underline{p},\underline{\partial})\,, \tag{60}$$

where $aX=a_\mu X^\mu$, with $a_\mu=t^A_\mu a_A$, a_A being an arbitrary constant c-number quantity. Then

$$\begin{aligned}
&\mathrm{e}^{-aX}\overline{f}\mathrm{e}^{aX}\\
&=\mathrm{e}^{-aX}\mathrm{e}^{-\frac{1}{2}\{\nabla,\partial\}}\mathrm{e}^{-pX}\hat{f}\mathrm{e}^{pX}\mathrm{e}^{\frac{1}{2}\{\nabla,\partial\}}\mathrm{e}^{aX}\\
&=\mathrm{e}^{-\frac{1}{2}\{\nabla,\partial\}}\mathrm{e}^{-a\partial}\mathrm{e}^{-aX}\mathrm{e}^{-pX}\hat{f}\mathrm{e}^{pX}\mathrm{e}^{aX}\mathrm{e}^{a\partial}\mathrm{e}^{\frac{1}{2}\{\nabla,\partial\}}\\
&=\mathrm{e}^{-\frac{1}{2}\{\nabla,\partial\}}\mathrm{e}^{-pX}\mathrm{e}^{-a\partial}\hat{f}\mathrm{e}^{a\partial}\mathrm{e}^{pX}\mathrm{e}^{\frac{1}{2}\{\nabla,\partial\}}\\
&=\overline{f},\qquad \hat{f}\in\mathcal{C}(\underline{p})\,.
\end{aligned} \tag{61}$$

In the second equality we use $[\nabla_\mu,aX]=a_\mu$ to move e^{aX} to the left, generating a factor $\mathrm{e}^{a\partial}$. Since $a_\mu\partial^\mu=a_A\partial^A$, this factor is a constant c-number and commutes with everything except p_A. In the last equality we use $\hat{f}$ not containing p_A.

iii) It is gauge and world covariant. This follows from using a covariant coordinate system such as Riemann normal coordinates at x_0. Because the covariant symbol is multiplicative (all x derivatives have already been taken) no quantities at points different from x_0 are needed.

iv) It preserves the hermiticity properties of the original operator. Assuming hermiticity rules in x space of the type

$$\begin{aligned}
&(\lambda X+\mu Y)^\dagger=\lambda^* X^\dagger+\mu^* Y^\dagger\,,\quad (XY)^\dagger=Y^\dagger X^\dagger\,,\\
&(x^\mu)^\dagger=+x^\mu\,,\quad (\nabla_\mu)^\dagger=-\nabla_\mu\,,\quad (Z_{\mu\nu})^\dagger=-Z_{\mu\nu}\,,\\
&(R_{\mu\nu}{}^\beta{}_\alpha)^\dagger=+R_{\mu\nu}{}^\beta{}_\alpha\,,\quad (T_{\mu\nu}{}^\lambda)^\dagger=+T_{\mu\nu}{}^\lambda\,,
\end{aligned} \tag{62}$$

etc., the hermitian character of an operator is shared by its covariant symbol by adding the prescriptions

$$(p_\mu)^\dagger=-p_\mu\,,\quad (\partial^\mu)^\dagger=+\partial^\mu \tag{63}$$

(recall that we are using a purely imaginary momentum variable throughout).

Of course, in practice the calculations implied in the definition of the covariant symbol cannot be carried out explicitly in full, a statement that also holds for the matrix element $\langle x|\hat{f}|x\rangle$ itself. A suitable approach compatible with its definition is to carry out a covariant derivative expansion of the symbol. In this counting each ∇_μ counts as first order, the torsion $T_{\mu\nu}{}^\lambda$ is also first order, $R_{\mu\nu}{}^\beta{}_\alpha$ is second order and so on, and p_μ and ∂^μ count as zeroth order.

A systematic computation is presented in the next section. By way of illustration we show here the covariant symbol of M and ∇_μ to second order in the derivative expansion for a general connection and general reference coordinate system. The result is expressed in terms of the tensors $P_{\mu_1\mu_2\cdots\mu_n}{}^\alpha{}_\beta$, which generalize that in (48) and (49):

$$[\nabla_{\mu_1},[\nabla_{\mu_2},\cdots,[\nabla_{\mu_n},p_\beta]\cdots]]=-P_{\mu_1\mu_2\cdots\mu_n}{}^\alpha{}_\beta p_\alpha\,. \tag{64}$$

This gives

$$\begin{aligned}
\overline{M}&=M-M_\mu\partial^\mu+\frac{1}{2}M_{\mu\nu}\partial^\mu\partial^\nu\\
&\quad+\frac{1}{2}M_\mu P_\nu{}^\mu{}_\lambda\partial^\lambda\partial^\nu+\mathcal{O}\left(\nabla^3\right),\\
\overline{\nabla}_\mu&=p_\mu+\frac{1}{2}P_\alpha{}^\lambda{}_\mu\{p_\lambda,\partial^\alpha\}-\frac{1}{4}\{Z_{\alpha\mu},\partial^\alpha\}\\
&\quad-\frac{1}{4}\left(P_{\alpha\beta}{}^\lambda{}_\mu+P_\alpha{}^\sigma{}_\beta P_\sigma{}^\lambda{}_\mu\right)\left\{p_\lambda,\partial^\alpha\partial^\beta\right\}+\mathcal{O}\left(\nabla^3\right).
\end{aligned} \tag{65}$$

We have used the property $P_\mu{}^\alpha{}_\nu-P_\nu{}^\alpha{}_\mu=T_{\mu\nu}{}^\alpha$. As we can see, $\overline{\nabla}_\mu$ is multiplicative in x space.

We can particularize these formulas to the case of normal coordinates at x_0 as RCS, but with arbitrary connection. Using the results of the next section (cf. (85) and (88)), one obtains at x_0[15]

$$\overline{M}=M-M_\mu\partial^\mu+\frac{1}{2}M_{\mu\nu}\partial^\mu\partial^\nu+\mathcal{O}\left(\nabla^3\right)\,, \tag{66}$$

$$\begin{aligned}
\overline{\nabla}_\mu&=p_\mu+\frac{1}{4}T_{\alpha\mu}{}^\lambda\{p_\lambda,\partial^\alpha\}-\frac{1}{4}\{Z_{\alpha\mu},\partial^\alpha\}\\
&\quad+\left(\frac{1}{12}R_{\mu\alpha}{}^\lambda{}_\beta-\frac{1}{6}T_{\alpha\beta\mu}{}^\lambda+\frac{1}{24}T_{\mu\alpha}{}^\sigma T_{\sigma\beta}{}^\lambda\right)\left\{p_\lambda,\partial^\alpha\partial^\beta\right\}\\
&\quad+\mathcal{O}\left(\nabla^3\right).
\end{aligned}$$

[14] The symbols X and Y are used to represent arbitrary operators. In particular X is unrelated to the vector field X^μ.

[15] We recall that only for multiplicative operators can one meaningfully take $x=x_0$.

This result is manifestly covariant and all operators involved are multiplicative with respect to x. The hermitian properties are explicit as well. Also note that $\overline{M}$ and $\overline{\nabla}_\mu$ are still operators with respect to the gauge and world indices. For instance, $Z_{\mu\nu}$ acts on world indices yielding the curvature tensor. It is also noteworthy that these formulas are formally independent of the domain of the operators involved. In particular, depending on the internal and world representation of the states, $Z_{\mu\nu}$ will act in a way or another. E.g., on a state which is gauge singlet and world scalar $|\phi\rangle$

$$Z_{\mu\nu}|\phi\rangle = 0\,, \tag{67}$$

yet

$$Z_{\mu\nu}p_\lambda|\phi\rangle = [Z_{\mu\nu},p_\lambda]|\phi\rangle + p_\lambda Z_{\mu\nu}|\phi\rangle = -R_{\mu\nu}{}^\sigma{}_\lambda\, p_\sigma|\phi\rangle\,. \tag{68}$$

When working with operators in $\mathcal{C}(Z)$, like $Z_{\mu\nu}$, one should be aware of seemingly paradoxical results. For example, for the same state $|\phi\rangle$ as before, and for B_λ a gauge singlet field,

$$0 = \langle\phi|Z_{\mu\nu}B_\lambda|\phi\rangle = -\langle\phi|R_{\mu\nu}{}^\sigma{}_\lambda\, B_\sigma|\phi\rangle \tag{69}$$

(using $\langle\phi|Z_{\mu\nu} = Z_{\mu\nu}|\phi\rangle = 0$). Indeed the result is zero, since the state $R_{\mu\nu}{}^\sigma{}_\lambda\, B_\sigma|\phi\rangle$ is a rank three tensor and cannot connect with the scalar state $|\phi\rangle$. There is also no contradiction if one uses instead the operator $Z_{\mu\nu}B^\mu C^\nu$ (again B^μ, C^ν gauge singlets) which is a scalar (and the previous argument would not apply), since in this case the operator $[Z_{\mu\nu}, B^\mu C^\nu]$ itself vanishes.

A preliminary definition of the covariant symbol was given in (55). As final definition we take (55) but using as RCS the Riemann normal coordinates associated to the given connection (these coordinates are defined for any connection), at each point x. That is, we use a different RCS at each point. This is perfectly well defined, since the operator is multiplicative and so equivalent to a function. For the same reason, the algebra homomorphism property is also not spoiled. There is an ambiguity in that normal coordinates at x are unique modulo a rigid general linear transformation. However, such an ambiguity does not reflect on the form of the covariant symbol when written in terms of p_μ and ∂^μ.

5 Computation of the covariant symbols

In this section we proceed to set up a systematic computation of the covariant symbols within a derivative expansion. The expansion is taken to fourth order.

5.1 Arbitrary reference coordinate system and arbitrary connection

Momentarily, we will work with an arbitrary RCS and arbitrary connection. The quantities t^A_μ, t^μ_A, p_A, p_μ, ∂^A and ∂^μ have already been defined and some of their properties noted in previous sections. Here we only note the equivalent definition

$$t^A_\mu := \left[\nabla_\mu, \xi^A\right], \tag{70}$$

where ξ^A is regarded as a c-number multiplicative operator. Next we introduce the world scalar operator (in $\mathcal{C}(\underline{W}, \underline{I}, \underline{p}, \underline{\partial})$)

$$\nabla_A := \frac{1}{2}\left\{\nabla_\mu, t^\mu_A\right\}, \tag{71}$$

which has the property

$$\nabla_\mu = \frac{1}{2}\left\{\nabla_A, t^A_\mu\right\}, \tag{72}$$

as is readily shown. For this and similar manipulations the following lemma is useful

Lemma: If the set of operators A_i satisfies

$$[A_i, A_j] = [[B, A_i], A_j] = 0 \quad \text{for all } i, j\,, \tag{73}$$

then

$$\frac{1}{4}\{\{B, A_i\}, A_j\} = \frac{1}{2}\{B, A_iA_j\} \quad \text{for all } i, j\,. \tag{74}$$

Applying the lemma, (72) follows from the definition (71) and the properties $[t^A_\mu, t^\mu_B] = [[\nabla_\mu, t^A_\nu], t^\alpha_B] = 0$. A crucial property of ∇_A is that $[\nabla_A, X]$ is multiplicative provided X is multiplicative, as is easily shown. As noted before, this property is not enjoyed by ∇_μ: for this case, the stronger assumption $X \in \mathcal{C}(\underline{\nabla}, \underline{Z})$ is needed.[16]

In the RCS $p_\mu X^\mu = p_A\xi^A$ holds, and also $\frac{1}{2}\{\nabla_\mu, \partial^\mu\}$ $= \nabla_A\partial^A$ (using (71) and the fact that ∂^A commutes with ∇_μ and t^μ_A). Therefore, we can reexpress the definition (55) of the covariant symbol in the form

$$\overline{f} = \mathrm{e}^{-\partial^A\nabla_A}\mathrm{e}^{-p_B\xi^B}\,\hat{f}\,\mathrm{e}^{p_C\xi^C}\mathrm{e}^{\partial^D\nabla_D}\,. \tag{75}$$

The commutation properties of the quantities ξ^A, ∇_A, p_A and ∂^A are as follows:

$$\begin{aligned}\left[\xi^A,\xi^B\right] &= \left[\xi^A,p_B\right] = \left[\xi^A,\partial_B\right] = \left[\nabla_A,p_B\right] = \left[\nabla_A,\partial^B\right]\\ &= [p_A,p_B] = \left[\partial^A,\partial^B\right] = 0\,,\\ \left[\nabla_A,\xi^B\right] &= \delta^B_A\,,\quad \left[\partial^A,p_B\right] = \delta^A_B\,,\quad [\nabla_A,\nabla_B] := Z_{AB}\,.\end{aligned} \tag{76}$$

We have introduced the operator Z_{AB}. Recursively we find

$$\begin{aligned}&Z_{A_1\cdots A_n} := [\nabla_{A_1}, Z_{A_2\cdots A_n}]\,,\\ &\left[Z_{A_1\cdots A_n},\xi^B\right] = [Z_{A_1\cdots A_n},,p_B] = \left[Z_{A_1\cdots A_n},\partial^B\right] = 0\,.\end{aligned} \tag{77}$$

[16] If ϕ is a c-number multiplicative operator and X is merely multiplicative, $[[\nabla_A, X], \phi] = [[\nabla_A, \phi], X] = 0$ since $[\nabla_A, \phi]$ is again a c-number multiplicative operator; however, $[\nabla_\mu, \phi]$ is not (it contains a world index).

In particular, note that the operators $Z_{A_1\cdots A_n}$ are multiplicative since they commute with ξ^A.

The important observation is that the commutation relations of ξ^A, ∇_A, p_A and ∂^A are identical to those of the flat case, as is also the definition of the covariant symbol in terms of these operators, (75). This immediately implies that the analogous of (21) hold

$$\begin{aligned}\overline{M} &= M - M_S\,\partial^S + \frac{1}{2!}M_{SS}\,(\partial^S)^2 - \frac{1}{3!}M_{SSS}\,(\partial^S)^3 + \cdots\,,\\ &\quad M\in\mathcal{C}(\underline{\nabla},\underline{Z},\underline{p},\underline{\partial})\\ \overline{\nabla}_A &= p_A - \frac{1}{2!}Z_{SA}\,\partial^S + \frac{2}{3!}Z_{SSA}\,(\partial^S)^2 - \frac{3}{4!}Z_{SSSA}\,(\partial^S)^3\\ &\quad + \cdots\,. \end{aligned} \tag{78}$$

with $M_A=[\nabla_A,M]$, $M_{AB}=[\nabla_A,[\nabla_B,M]]$, and so on, and S standing for contracted symmetrized indices of the type A, B, …, (cf. Appendix A). (Actually, the equation for $\overline{M}$ holds too for $M\in\mathcal{C}(\underline{\nabla},\underline{p},\underline{\partial})$.)

Unfortunately, this simple result is not sufficient. We need $\overline{\nabla}_\mu$ instead of $\overline{\nabla}_A$, and quantities formed with M and ∇_μ instead of ∇_A and $Z_{A_1,A_2,\dots,A_n}$.

5.2 Riemann normal coordinates and arbitrary connection

Let us consider first $\overline{M}$. Since $M\in\mathcal{C}(\underline{Z})$, M commutes with t^μ_A and t^A_μ and their derivatives. Hence, we find

$$\begin{aligned} M_A &= M_\mu\,t^\mu_A\,,\\ M_{AB} &= M_{\mu\nu}\,t^\mu_A\,t^\nu_B + M_\nu\,t^\nu_{\mu B}\,t^\mu_A\,, \end{aligned} \tag{79}$$

and so on. As before, we use the notation

$$t^\lambda_{\mu_1\dots\mu_n A} = \left[\nabla_{\mu_1}, t^\lambda_{\mu_2\dots\mu_n A}\right],\quad t^A_{\mu_1\dots\mu_n\lambda} = \left[\nabla_{\mu_1}, t^A_{\mu_2\dots\mu_n\lambda}\right]. \tag{80}$$

The derivatives of the type $t^\lambda_{\mu_1\dots\mu_n A}$ can be expressed in terms of t^λ_B and $t^B_{\mu_1\dots\mu_n\lambda}$ using the relations (35), e.g.

$$t^\nu_{\mu A} = -t^\nu_B\,t^\lambda_A\,t^B_{\mu\lambda}\,. \tag{81}$$

This gives for the term with two derivatives in $\overline{M}$

$$M_{SS} = \left(M_{ss} + M_\lambda\,t^A_{ss}\,t^\lambda_A\right)t^s_S\,t^s_S\,. \tag{82}$$

To proceed further we make a choice of RCS (for given base point x_0), namely, we choose the usual Riemann normal coordinates at x_0. These are the coordinates such that the curves $\xi^A(t)=tv^A$ are geodesics passing through x_0, the geodesics being the straightest lines with respect to the given connection. A practical equivalent definition is to take the coordinates ξ^A so that

$$\xi^A(x_0)=0\,,\qquad t^A_{\underbrace{ss\dots s}_{n}}\Big|_{x_0}=0\qquad\text{for }\ n\geq 2\,. \tag{83}$$

(This means that, for $n\geq 2$, the completely symmetrized component of $t^A_{\mu_1\dots\mu_n}$ vanishes.) In this form the normal coordinates system was used in a similar context in [53]. Given x_0 and t^A_μ at x_0, the normal coordinates are locally unique, since the $t^A_{\mu_1\dots\mu_n\lambda}$ at x_0 can be obtained recursively in terms of the curvature and torsion tensors and their derivatives, using the definition above. For instance,

$$t^A_{\mu\nu} = \nabla_\mu\nabla_\nu\xi^A = t^A_{\nu\mu} - T_{\mu\nu}{}^\lambda\,t^A_\lambda\,, \tag{84}$$

and using (83) (i.e., $t^A_{\mu\nu}$ is purely antisymmetric at x_0)

$$t^A_{\mu\nu} = -\frac{1}{2}T_{\mu\nu}{}^\lambda\,t^A_\lambda\,,\qquad\text{at } x_0\,. \tag{85}$$

Likewise, starting from

$$t^A_{\mu\nu\alpha} + \text{five permutations} = 0\,,\qquad\text{at } x_0\,, \tag{86}$$

and using the identity

$$t^A_{\alpha\beta\gamma} = t^A_{\beta\alpha\gamma} - R_{\alpha\beta}{}^\lambda{}_\gamma\,t^A_\lambda - T_{\alpha\beta}{}^\lambda\,t^A_{\lambda\gamma} \tag{87}$$

(plus derivatives of (84)) to bring the five permutations to coincide with the first ordering, gives $t^A_{\mu\nu\alpha}$ at x_0. This yields

$$\begin{aligned} t^A_{\alpha\beta\gamma} = \Big(&-\frac{1}{3}R_{\alpha\beta}{}^\lambda{}_\gamma + \frac{1}{3}R_{\gamma\alpha}{}^\lambda{}_\beta - \frac{1}{2}T_{\alpha\beta\gamma}{}^\lambda\\ &+\frac{1}{6}T_{\beta\gamma\alpha}{}^\lambda - \frac{1}{6}T_{\gamma\alpha\beta}{}^\lambda - \frac{1}{4}T_{\beta\gamma}{}^\sigma T_{\sigma\alpha}{}^\lambda\\ &-\frac{1}{12}T_{\gamma\alpha}{}^\sigma T_{\sigma\beta}{}^\lambda + \frac{1}{12}T_{\alpha\beta}{}^\sigma T_{\sigma\gamma}{}^\lambda\Big)t^A_\lambda\,,\qquad\text{at } x_0\,. \end{aligned} \tag{88}$$

The same technique applies for computing any higher derivative of ξ^A at x_0.

Let us come back now to the evaluation of M_{SS}. Clearly for normal coordinates (but arbitrary connection), (82) reduces to $M_{SS}=M_{ss}t^s_S\,t^s_S$ at the origin. In fact a similar reduction happens to all orders, that is,[17]

$$M_{S\dots S} = M_{s\dots s}\,t^s_S\cdots t^s_S\,,\qquad\text{at } x_0\,. \tag{89}$$

In summary, the covariant symbol of M, to all orders in the derivative expansion, is given by an expression fully analogous to that of the purely gauge case, namely,

$$\begin{aligned}\overline{M} &= M - M_s\,\partial^s + \frac{1}{2!}M_{ss}\,(\partial^s)^2 - \frac{1}{3!}M_{sss}\,(\partial^s)^3 + \cdots\,,\\ &\quad M\in\mathcal{C}(\underline{\nabla},\underline{Z},\underline{p},\underline{\partial})\,,\end{aligned} \tag{90}$$

[17] This can be seen recursively as follows. $M_{S\dots S}$ contains a first term $M_{s\dots s}\,t^s_S\cdots t^s_S$ plus other terms containing a factor $t^A_{s\dots s}$ (with two or more s). Now, in the present case (namely, $M\in\mathcal{C}(\underline{\nabla},\underline{Z},\underline{p},\underline{\partial})$ but not, e.g., for $Z_{\mu\nu}$), each new ∇_S is equivalent to $t^s_S\nabla_s$. If ∇_s acts on a factor t^s_S in the first term, the formula

$$t^\lambda_{sS} = -t^\lambda_A\,t^s_S\,t^A_{ss}$$

applies and this vanishes at x_0. Likewise, if ∇_s acts on the other terms, there will always remain a $t^A_{s\dots s}$ factor with two or more s. So the only surviving piece at x_0 is that obtained by ∇_s acting on $M_{s\dots s}$. This yields (89). See footnote 20 for an alternative proof.

where s are symmetrized world indices. Note that, unlike (78), this expression does not apply for arbitrary operators in $\mathcal{C}(\underline{\nabla}, \underline{p}, \underline{\partial})$ such as $Z_{\mu\nu}$.

Next, we need to introduce world-index counterparts of $Z_{A_1\ldots A_n}$. As discussed in Sect. 3, the objects $[\nabla_{\mu_1}, [\cdots, \nabla_{\mu_n}]\cdots]$ are not in general multiplicative operators. Instead, we recursively define

$$\begin{aligned} Z_{\mu\nu} &= [\nabla_\mu, \nabla_\nu] + \frac{1}{2}\left\{\nabla_\lambda, T_{\mu\nu}{}^\lambda\right\}, \\ Z_{\alpha\mu\nu} &= [\nabla_\alpha, Z_{\mu\nu}] - \frac{1}{2}\left\{\nabla_\lambda, R_{\mu\nu}{}^\lambda{}_\alpha\right\}, \\ &\vdots \\ Z_{\alpha\mu_1\cdots\mu_n} &= [\nabla_\alpha, Z_{\mu_1\cdots\mu_n}] - \frac{1}{2}\left\{\nabla_\lambda, R_{\mu_1\cdots\mu_n}{}^\lambda{}_\alpha\right\}, \end{aligned} \tag{91}$$

(with $R_{\sigma\mu\nu}{}^\alpha{}_\beta := \nabla_\sigma R_{\mu\nu}{}^\alpha{}_\beta$, etc.). These operators are multiplicative, and indeed for a gauge singlet V^σ they satisfy

$$\begin{aligned} [Z_{\mu\nu}, V^\sigma] &= R_{\mu\nu}{}^\sigma{}_\lambda V^\lambda, \\ [Z_{\alpha\mu\nu}, V^\sigma] &= R_{\alpha\mu\nu}{}^\sigma{}_\lambda V^\lambda, \\ &\vdots \qquad\qquad V \in \mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{I}), \\ [Z_{\mu_1\cdots\mu_n}, V^\sigma] &= R_{\mu_1\cdots\mu_n}{}^\sigma{}_\lambda V^\lambda \end{aligned} \tag{92}$$

(and of course a similar action on each world index in the case of tensors). In addition they are antihermitian. In terms of these, one obtains the following relations at x_0:[18]

$$\begin{aligned} Z_{AB} &= \frac{1}{2}\left\{t^\alpha_A\, t^\beta_B, Z_{\alpha\beta}\right\} \qquad\qquad \text{(all at } x_0\text{)}, \\ Z_{ABC} &= \frac{1}{2}\Big\{t^\alpha_A\, t^\beta_B\, t^\mu_C, Z_{\alpha\beta\mu} + \frac{1}{4}\left\{Z_{\beta\lambda}, T_{\alpha\mu}{}^\lambda\right\} \\ &\quad - \frac{1}{4}\left\{Z_{\mu\lambda}, T_{\alpha\beta}{}^\lambda\right\}\Big\}. \end{aligned} \tag{93}$$

We can proceed now to the evaluation of $\overline{\nabla}_\mu$. To do this we use the relation

$$\overline{\nabla}_\mu = \frac{1}{2}\left\{\overline{\nabla}_A, \overline{t}^A_\mu\right\}, \tag{94}$$

which follows from (72) and the homomorphism property of the covariant symbols. The quantity $\overline{t}^A_\mu$ is easily obtained to second order using $M = t^A_\mu$ in (90) and the formulas (85) and (88):

$$\begin{aligned} \overline{t}^A_\mu = t^A_\lambda \Big[&\delta^\lambda_\mu + \frac{1}{2} T_{s\mu}{}^\lambda\, \partial^s \\ &+ \left(\frac{1}{6} R_{\mu s}{}^\lambda{}_s - \frac{1}{3} T_{ss\mu}{}^\lambda + \frac{1}{12} T_{s\mu}{}^\sigma T_{s\sigma}{}^\lambda\right)(\partial^s)^2 \\ &+ \mathcal{O}\left(\nabla^3\right)\Big]. \end{aligned} \tag{95}$$

On the other hand $\overline{\overline{\nabla}}_A$ is obtained to second order from (78) and the first equation in (93). In this way we reproduce the result for $\overline{\overline{\nabla}}_\mu$ in (66).

5.3 Riemannian connection results

From now on we restrict ourselves to the Riemannian connection, since the absence of torsion considerably simplifies the expressions.

For the Riemannian connection one finds at x_0

$$\begin{aligned} t^A_{\mu\nu} &= 0, \\ t^A_{\alpha\mu\nu} &= \frac{1}{3}\left(R^\lambda{}_{\mu\nu\alpha} + R^\lambda{}_{\nu\mu\alpha}\right) t^A_\lambda, \qquad \text{at } x_0. \end{aligned} \tag{96}$$

Some higher order results needed to obtain $\overline{t}^A_\mu$ are as follows:

$$\begin{aligned} t^A_{s\mu} &= 0, \\ t^A_{ss\mu} &= \frac{1}{3} R_{\mu s}{}^\lambda{}_s\, t^A_\lambda, \\ t^A_{sss\mu} &= \frac{1}{2} R_{s\mu s}{}^\lambda{}_s\, t^A_\lambda, \\ t^A_{ssss\mu} &= \left[\frac{3}{5} R_{ss\mu s}{}^\lambda{}_s + \frac{7}{15} R_{\mu s}{}^\sigma{}_s R_{\sigma s}{}^\lambda{}_s\right] t^A_\lambda, \qquad \text{all at } x_0. \end{aligned} \tag{97}$$

With (90) this gives

$$\begin{aligned} \overline{t}^A_\mu = t^A_\lambda \Big[&\delta^\lambda_\mu + \frac{1}{6} R_{\mu s}{}^\lambda{}_s\, (\partial^s)^2 - \frac{1}{12} R_{s\mu s}{}^\lambda{}_s\, (\partial^s)^3 \\ &+ \left(\frac{1}{40} R_{ss\mu s}{}^\lambda{}_s + \frac{7}{360} R_{\mu s}{}^\sigma{}_s R_{\sigma s}{}^\lambda{}_s\right)(\partial^s)^4 + \mathcal{O}(\nabla^5)\Big]. \end{aligned} \tag{98}$$

In addition, for the Riemannian connection one has the Bianchi identities

$$0 = Z_{\alpha_1\cdots\alpha_n\mu_1\mu_2\mu_3} + Z_{\alpha_1\cdots\alpha_n\mu_2\mu_3\mu_1} + Z_{\alpha_1\cdots\alpha_n\mu_3\mu_1\mu_2}, \tag{99}$$

and the following relations (which hold at x_0):

$$\begin{aligned} Z_{AB} &= \frac{1}{2}\left\{t^\alpha_A\, t^\beta_B, Z_{\alpha\beta}\right\}, \\ Z_{ABC} &= \frac{1}{2}\left\{t^\alpha_A\, t^\beta_B\, t^\mu_C, Z_{\alpha\beta\mu}\right\}, \qquad\qquad \text{(all at } x_0\text{)} \end{aligned} \tag{100}$$

[18] Naturally, instead of (91), we could have adopted a definition of the type

$$Z'_{\mu_1\cdots\mu_n} = \frac{1}{2}\left\{t^{A_1}_{\mu_1}\cdots t^{A_n}_{\mu_n}, Z_{A_1\cdots A_n}\right\},$$

which has all the good properties, and in particular

$$Z_{A_1\cdots A_n} = \frac{1}{2}\left\{t^{\mu_1}_{A_1}\cdots t^{\mu_n}_{A_n}, Z'_{\mu_1\cdots\mu_n}\right\}.$$

However, the relations similar to (92) become more complicated. The definition adopted corresponds to $Z_{\alpha\mu_1\ldots\mu_n} = \frac{1}{2}\{t^A_\alpha, [\nabla_A, Z_{\mu_1\ldots\mu_n}]\}$.

$$Z_{ABCD} = \frac{1}{2}\Big\{ t^\alpha_A\, t^\beta_B\, t^\mu_C\, t^\nu_D, Z_{\alpha\beta\mu\nu} + \frac{1}{6}\left\{ Z_{\lambda\mu}, R^\lambda{}_{\beta\nu\alpha} + R^\lambda{}_{\nu\beta\alpha} \right\} - \frac{1}{6}\left\{ Z_{\lambda\nu}, R^\lambda{}_{\beta\mu\alpha} + R^\lambda{}_{\mu\beta\alpha} \right\} \Big\}.$$

Combining all the previous relations and (94) we can now write down the expression for $\overline{\nabla}_\mu$, which we have computed to four derivatives

$$\overline{\nabla}_\mu = \overline{\nabla}_\mu{}^{(0)} + \overline{\nabla}_\mu{}^{(1)} + \overline{\nabla}_\mu{}^{(2)} + \overline{\nabla}_\mu{}^{(3)} + \overline{\nabla}_\mu{}^{(4)} + \mathcal{O}(\nabla^5)\,. \tag{101}$$

The result is as follows:

$$\begin{aligned}
\overline{\nabla}_\mu{}^{(0)} &= p_\mu\,,\\
\overline{\nabla}_\mu{}^{(1)} &= 0\,,\\
\overline{\nabla}_\mu{}^{(2)} &= -\frac{1}{4}\left\{ Z_{s\mu}, \partial^s \right\} + \frac{1}{12}\left\{ [Z_{s\mu}, p_s], (\partial^s)^2 \right\}\,,\\
\overline{\nabla}_\mu{}^{(3)} &= \frac{1}{6}\left\{ Z_{ss\mu}, (\partial^s)^2 \right\} - \frac{1}{24}\left\{ [Z_{ss\mu}, p_s], (\partial^s)^3 \right\}\,,\\
\overline{\nabla}_\mu{}^{(4)} &= -\frac{1}{16}\left\{ Z_{sss\mu}, (\partial^s)^3 \right\} + \frac{1}{80}\left\{ [Z_{sss\mu}, p_s], (\partial^s)^4 \right\}\\
&\quad + \frac{1}{48}\left\{ Z_{s\lambda}, [Z_{s\mu}, \partial^\lambda](\partial^s)^2 \right\}\\
&\quad - \frac{7}{720}\left\{ [Z_{s\lambda}, p_s], [Z_{s\mu}, \partial^\lambda](\partial^s)^3 \right\}\,.
\end{aligned} \tag{102}$$

The result has been written in a manifestly antihermitian form. Although the Riemann tensor does not appear, these formulas hold only for the Riemann connection. The formulas have been verified in various ways. In particular, for $M \in \mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{p}, \underline{\partial})$ one can apply the expansion (90) to M and also to $\overline{M}_\mu = [\nabla_\mu, M]$ since it falls in the same class. Then one can check that the formulas preserve the homomorphism property, in the form

$$\overline{M}_\mu = \left[\overline{\nabla}_\mu, \overline{M} \right]\,. \tag{103}$$

Another check comes from computing the covariant symbol of $Z_{\mu\nu}$ (for which (90) does not apply). A direct computation to four derivatives gives

$$\overline{Z}_{\mu\nu} = Z_{\mu\nu} - \frac{1}{2}\left\{ Z_{s\mu\nu}, \partial^s \right\} + \frac{1}{4}\left\{ Z_{ss\mu\nu}, (\partial^s)^2 \right\} + \mathcal{O}\left(\nabla^5\right). \tag{104}$$

As we have verified, this expression satisfies

$$\overline{Z}_{\mu\nu} = \left[\overline{\nabla}_\mu, \overline{\nabla}_\nu \right]\,. \tag{105}$$

These checks would serve to determine some of the coefficients in the expression of $\overline{\nabla}_\mu$, but not all.

Another operator of great interest in applications is the Laplacian,

$$\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{106}$$

Since $[\nabla_\lambda, g_{\mu\nu}] = 0$, an application of (90) gives

$$\overline{g}_{\mu\nu} = g_{\mu\nu}\,. \tag{107}$$

(Also clear from the definition (55) since $g_{\mu\nu}$ commutes with all operators there.) Therefore we can use our previous results to obtain the covariant symbol of the Laplacian, by means of

$$\overline{\Delta} = g^{\mu\nu}\overline{\nabla}_\mu\overline{\nabla}_\nu\,. \tag{108}$$

This yields an expansion of $\overline{\Delta}$ to four derivatives,

$$\overline{\Delta} = \overline{\Delta}^{(0)} + \overline{\Delta}^{(1)} + \overline{\Delta}^{(2)} + \overline{\Delta}^{(3)} + \overline{\Delta}^{(4)} + \mathcal{O}(\nabla^5)\,, \tag{109}$$

with

$$\begin{aligned}
\overline{\Delta}^{(0)} &= p_\mu p^\mu\,,\\
\overline{\Delta}^{(1)} &= 0\,,\\
\overline{\Delta}^{(2)} &= -\frac{1}{2}\left\{ Z_{s\mu}, p^\mu\partial^s \right\} + \frac{1}{3}\left[[Z_{s\mu}, p^\mu], \partial^s \right]\\
&\quad + \frac{1}{6}\left\{ [Z_{s\mu}, p_s]p^\mu, (\partial^s)^2 \right\}\,,\\
\overline{\Delta}^{(3)} &= \frac{1}{6}\left\{ Z_{ss\mu}, \left\{ p^\mu, (\partial^s)^2 \right\} \right\} - \frac{2}{3}\left[Z^\mu{}_{s\mu}, \partial^s \right]\\
&\quad - \frac{1}{12}\left\{ [Z_{ss\mu}, p_s]p^\mu, (\partial^s)^3 \right\}\,,\\
\overline{\Delta}^{(4)} &= -\frac{1}{16}\left\{ Z_{sss\mu}, \left\{ p^\mu, (\partial^s)^3 \right\} \right\}\\
&\quad + \frac{1}{40}\left\{ [Z_{sss\mu}, p_s]p^\mu, (\partial^s)^4 \right\}\\
&\quad - \frac{1}{16}\left\{ Z_{s\mu}, \left\{ \left[Z_s{}^\mu, p_s \right], (\partial^s)^3 \right\} \right\} + \frac{1}{8}\left\{ Z_{s\mu}Z_s{}^\mu, (\partial^s)^2 \right\}\\
&\quad + \frac{1}{30}\left\{ [Z_{s\mu}, p_s][Z_s{}^\mu, p_s], (\partial^s)^4 \right\}\\
&\quad + \frac{1}{60}\left[Z^\mu{}_\alpha, \partial^\alpha \right]\left[Z_{\mu\beta}, \partial^\beta \right]\\
&\quad + \frac{2}{45}\left[Z^\mu{}_\alpha, \partial^\beta \right]\left[Z_{\mu\beta}, \partial^\alpha \right] + \frac{2}{45}\left[Z^\mu{}_\alpha, \partial^\beta \right]\left[Z_\mu{}^\alpha, \partial_\beta \right]\\
&\quad + \frac{1}{3}\left[Z_{s\mu\nu}{}^\mu, \partial^\nu \right]\partial^s + \frac{1}{60}\left[Z_{\mu s\nu}{}^\mu, \partial^\nu \right]\partial^s\\
&\quad - \frac{1}{40}\left[Z_\mu{}^\mu{}_{\nu s}, \partial^\nu \right]\partial^s\,.
\end{aligned} \tag{110}$$

Once again the result has been written in an explicit hermitian form. It is noteworthy that in all terms the metric has been used once to raise one index (as in the Laplacian itself), except in the second term with coefficient $2/45$ in $\overline{\Delta}^{(4)}$, in which the metric is used thrice. This is because specific properties of the Riemann connection have been used in simplifying the formula. (The metric has to appear an odd number of times since the Laplacian is odd under $g_{\mu\nu} \to -g_{\mu\nu}$ whereas the connection itself is even.)

The previous formulas can also be brought to a more systematic or "standard" form. We define such a standard form by the requirement that the quantities R, Z, p and ∂ appear in the expressions in this very order (i.e., the ∂ occupy the rightmost position, then the p, and so on.) Of course, in standard form hermiticity is no longer manifest. For the covariant symbol of the covariant derivative, we

obtain

$$\begin{aligned}
\overline{\nabla}_\mu{}^{(2)} &= \frac{1}{2}Z_{\mu s}\partial^s + \frac{1}{6}\mathcal{R}_{\mu s}\,\partial^s + \frac{1}{6}R_{\mu s}{}^\alpha{}_s\,p_\alpha(\partial^s)^2\,,\\
\overline{\nabla}_\mu{}^{(3)} &= \frac{1}{3}Z_{ss\mu}(\partial^s)^2 + \frac{1}{8}R_{\alpha\mu s}{}^\alpha{}_s(\partial^s)^2 - \frac{1}{8}\mathcal{R}_{ss\mu}(\partial^s)^2\\
&\quad - \frac{1}{12}R_{s\mu s}{}^\alpha{}_s p_\alpha(\partial^s)^3\,,\\
\overline{\nabla}_\mu{}^{(4)} &= -\frac{1}{8}Z_{sss\mu}(\partial^s)^3 + \frac{1}{24}R_{\mu s}{}^\alpha{}_s Z_{\alpha s}(\partial^s)^3\\
&\quad + \frac{1}{20}\mathcal{R}_{sss\mu}(\partial^s)^3 - \frac{1}{10}R_{s\alpha\mu s}{}^\alpha{}_s(\partial^s)^3\\
&\quad - \frac{13}{720}\mathcal{R}_{s\alpha}R_{\mu s}{}^\alpha{}_s(\partial^s)^3 + \frac{1}{90}R_{\alpha\mu}{}^\beta{}_s\,R_{\beta s}{}^\alpha{}_s(\partial^s)^3\\
&\quad + \frac{1}{40}R_{ss\mu s}{}^\alpha{}_s\,p_\alpha(\partial^s)^4 + \frac{7}{360}R_{\mu s}{}^\alpha{}_s\,R_{\alpha s}{}^\beta{}_s\,p_\beta(\partial^s)^4\,.
\end{aligned} \tag{111}$$

In the same way, for the covariant symbol of the Laplacian,

$$\begin{aligned}
\overline{\Delta}^{(2)} &= \frac{1}{6}\mathbf{R} + Z^\alpha{}_s\,p_\alpha\partial^s - \frac{1}{3}\mathcal{R}^\alpha{}_s\,p_\alpha\partial^s + \frac{1}{3}R^\alpha{}_s{}^\beta{}_s\,p_\alpha p_\beta(\partial^s)^2\,,\\
\overline{\Delta}^{(3)} &= -\frac{1}{3}Z_\alpha{}^\alpha{}_s\partial^s - \frac{1}{4}\mathbf{R}_s\,\partial^s\\
&\quad + \frac{2}{3}Z_{ss\alpha}\,p^\alpha(\partial^s)^2 + \frac{1}{6}\mathcal{R}_{ss\alpha}\,p^\alpha(\partial^s)^2\\
&\quad + \frac{1}{6}R^\alpha{}_{\alpha s}{}^\beta{}_s\,p_\beta(\partial^s)^2 - \frac{1}{6}R_s{}^\alpha{}_s{}^\beta\,p_\alpha p_\beta(\partial^s)^3\,,\\
\overline{\Delta}^{(4)} &= \frac{3}{20}\mathbf{R}_{ss}(\partial^s)^2 - \frac{1}{10}R^\alpha{}_{\beta\alpha s}{}^\beta{}_s(\partial^s)^2 + \frac{1}{4}Z_s{}^\alpha{}_{\alpha s}(\partial^s)^2\\
&\quad + \frac{29}{120}\mathcal{R}^\alpha{}_s\mathcal{R}_{\alpha s}(\partial^s)^2 + \frac{1}{4}\mathcal{R}^\alpha{}_s\,Z_{\alpha s}(\partial^s)^2\\
&\quad - \frac{31}{120}\mathcal{R}^\alpha{}_\beta R_{\alpha s}{}^\beta{}_s(\partial^s)^2 - \frac{1}{60}R_s{}^{\alpha\beta}{}_\gamma\,R_{s\alpha}{}^\gamma{}_\beta(\partial^s)^2\\
&\quad + \frac{1}{4}Z^\alpha{}_s\,Z_{\alpha s}(\partial^s)^2 - \frac{1}{20}\mathcal{R}_{sss\alpha}\,p^\alpha(\partial^s)^3\\
&\quad - \frac{3}{20}R_s{}^\alpha{}_{\alpha s}{}^\beta{}_s\,p_\beta(\partial^s)^3 - \frac{1}{4}Z_{sss\alpha}\,p^\alpha(\partial^s)^3\\
&\quad + \frac{13}{120}\mathcal{R}^\alpha{}_s\,R_{\alpha s}{}^\beta{}_s\,p_\beta(\partial^s)^3 - \frac{1}{15}R^{\alpha\beta}{}_s{}_s\,R_{s\alpha}{}^\gamma{}_\beta\,p_\gamma(\partial^s)^3\\
&\quad + \frac{1}{4}R^{\alpha\beta}{}_s{}_s\,Z_{\alpha s}\,p_\beta(\partial^s)^3 + \frac{1}{20}R_{ss}{}^{\alpha\beta}{}_s\,p_\alpha p_\beta(\partial^s)^4\\
&\quad + \frac{1}{15}R^{\alpha\gamma}{}_s{}_s R_{\gamma s}{}^\beta{}_s\,p_\alpha p_\beta(\partial^s)^4\,.
\end{aligned} \tag{112}$$

In this alternative form the metric appears exactly once in each term.

Equations (90), (111) and (112) are the main result of this work. They extend the results of Pletnev and Banin to curved space-time and can be used immediately to compute diagonal matrix elements by means of (58). Obvious applications are the computation of the heat kernel in the non-minimal case within a strict covariant derivative expansion. Such a calculation has been carried out, both for traced and untraced coefficients, and it will be presented elsewhere. Another interesting application is to the computation of the effective action of fermions with chiral gauge and curvature connections. This type of calculation has been done in the flat space-time case within a covariant derivative expansion for both the normal and abnormal parity components of the effective action in [48, 50]. So it would seem natural to extend such results to the case of curved space-time.

6 Sample computation using covariant symbols

For the purposes of illustration, in this section we apply the method of covariant symbols to the computation of the diagonal matrix element of a concrete operator. In Appendix B we carry out the analogous computation using the method of symbols. As operator we take

$$\hat{\mathcal{Q}}_{\mu\nu} = \nabla_\mu \frac{1}{m^2 - \Delta}\nabla_\nu\,, \tag{113}$$

where m is a positive constant c-number. The operator is defined on a d-dimensional Euclidean space-time. d is kept arbitrary so that ultraviolet convergence of the matrix element is assured in the sense of dimensional regularization. Note that, through a standard functional transform, the operator can be related to

$$\hat{\mathcal{H}}_{\mu\nu} = \nabla_\mu \mathrm{e}^{\tau\Delta}\nabla_\nu \tag{114}$$

which is well behaved in the ultraviolet (for positive τ).

The covariant derivative ∇_μ includes gauge and world connections, the latter being the Riemannian connection. We do not specify the gauge connection and also the space of states is kept unspecified. In particular, the states may have any tensorial structure. Also the operator itself is not a world scalar.

We have chosen $\mathcal{Q}_{\mu\nu}$ instead of $(m^2 - \Delta)^{-1}$ or $\mathrm{e}^{\tau\Delta}$ in order to illustrate the method with an operator that cannot be obtained as a variation of the heat kernel, for which many results and alternative procedures are available.

Specifically, we will compute

$$\mathcal{Q}_{\mu\nu}(x) = \langle x|\hat{\mathcal{Q}}_{\mu\nu}|x\rangle \tag{115}$$

through second derivatives, i.e., neglecting terms with four or more covariant derivatives. Because m^2 is a constant, in this case the derivative expansion is equivalent to an inverse mass expansion. The terms neglected introduce a relative error $\mathcal{O}(1/m^4)$.

Using the relation (56), we can write

$$\mathcal{Q}_{\mu\nu}(x) = \left\langle \overline{\mathcal{Q}}_{\mu\nu} \right\rangle\,, \tag{116}$$

where $\overline{\mathcal{Q}}_{\mu\nu}$ is the covariant symbol of $\hat{\mathcal{Q}}_{\mu\nu}$, and we have introduced the notation (X representing an arbitrary multiplicative quantity here)

$$\langle X\rangle := \quad \frac{1}{\sqrt{g^{(\xi)}(x)}}\int \frac{\mathrm{d}^d p_A}{(2\pi)^d}\,\langle x|X|0\rangle\,, \qquad X \in \mathcal{C}(\underline{\nabla})\,. \tag{117}$$

Using the homomorphism property of the covariant symbol implies

$$\mathcal{Q}_{\mu\nu}(x) = \left\langle \overline{\nabla}_\mu \frac{1}{m^2 - \overline{\Delta}}\overline{\nabla}_\nu \right\rangle\,. \tag{118}$$

Next we proceed to substitute the derivative expansion expressions of $\overline{\nabla}_\mu$ and $\overline{\Delta}$. A simplification occurs by noting that $\overline{\nabla}_\lambda$ differs from p_λ only by terms with ∂^σ. As can be seen in (111), $\overline{\nabla}_\mu{}^{(n)}$ for $n \geq 1$ has ∂^σ at the right, or also at the left since these quantities are antihermitian. Thus

$$\left\langle X\overline{\nabla}_\lambda \right\rangle = \langle X p_\lambda \rangle \,, \qquad \left\langle \overline{\nabla}_\lambda X \right\rangle = \langle p_\lambda X \rangle \,. \tag{119}$$

This gives

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) &= \left\langle p_\mu \left[m^2 - p_\lambda p^\lambda - \overline{\Delta}^{(2)} \right]^{-1} p_\nu \right\rangle + \mathcal{O}\left(\nabla^4\right) \\ &= \left\langle p_\mu \left[N + N\left(\frac{1}{6}\mathbf{R} + \left(Z_{\alpha\beta} - \frac{1}{3}\mathcal{R}_{\alpha\beta} \right) p^\alpha \partial^\beta \right.\right.\right. \\ &\quad \left.\left.\left. + \frac{1}{3} R_{\alpha\lambda\beta\sigma}\, p^\alpha p^\beta \partial^\lambda \partial^\sigma \right) N \right] p_\nu \right\rangle + \mathcal{O}\left(\nabla^4\right) \,,\end{aligned} \tag{120}$$

where

$$N := \frac{1}{m^2 + p^2} \,, \quad p^2 := -p_\lambda p^\lambda \geq 0 \,. \tag{121}$$

The momentum derivatives are easily computed using the identities

$$\begin{aligned}[\partial^\alpha, N] &= 2p^\alpha N^2 \\ \left\langle X \partial^\beta N p_\nu \right\rangle &= \left\langle X \left(2p^\beta p_\nu N^2 + \delta^\beta_\nu N \right) \right\rangle \\ \left\langle X \partial^\alpha \partial^\beta N p_\nu \right\rangle &= \left\langle X \big(2 \left(\delta^\alpha_\nu p^\beta + \delta^\beta_\nu p^\alpha + g^{\alpha\beta} p_\nu \right) N^2 \right. \\ &\quad \left. + 8 p^\alpha p^\beta p_\nu N^3 \big) \right\rangle \,.\end{aligned} \tag{122}$$

(Of course, one can choose to apply ∂^μ to the left, by parts.) In addition we group together the p and the N using

$$p_\mu Z_{\alpha\beta} = Z_{\alpha\beta}\, p_\mu + R_{\alpha\beta}{}^\lambda{}_\mu\, p_\lambda \,, \qquad [Z_{\mu\nu}, N] = 0 \,. \tag{123}$$

This produces[19]

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle & p_\mu p_\nu N + \frac{1}{6}\mathbf{R}\, p_\mu p_\nu N^2 \\ & + \left(Z^\alpha_{\ \nu} - \frac{1}{3}\mathcal{R}^\alpha_{\ \nu} \right) p_\alpha p_\mu N^2 \\ & + R^{\alpha\ \ \beta}_{\ \mu\ \ \nu}\, p_\alpha p_\beta N^2 \Big\rangle + \mathcal{O}\left(\nabla^4\right) \,.\end{aligned} \tag{124}$$

The momentum integrals can already be taken. In principle, for an expression $\langle X \rangle$ (X being a multiplicative operator) the proper procedure would be to take matrix elements $\langle x|X|0\rangle$, and then proceed to carry out the integration over the d constants p_A. In the present case, all the p-dependence is contained in blocks of the type $(p_{\mu_1} \cdots p_{\mu_{2r}} N^n)$, where all quantities commute among them (all p have been put together in each term). Thus in this case we can equivalently carry out the momentum integration in each block using

$$\begin{aligned}\left\langle p_{\mu_1} \cdots p_{\mu_{2r}} N^n \right\rangle_p &:= \frac{1}{\sqrt{g^{(\xi)}(x)}} \int \frac{\mathrm{d}^d p_A}{(2\pi)^d}\, p_{\mu_1} \cdots p_{\mu_{2r}} N^n \\ &= \frac{1}{\sqrt{g(x)}} \int \frac{\mathrm{d}^d p_\mu}{(2\pi)^d}\, p_{\mu_1} \cdots p_{\mu_{2r}} N^n \,.\end{aligned} \tag{125}$$

In addition, we can introduce the multiplicative operator $\hat{\mathcal{Q}}'_{\mu\nu}$ such that

$$\mathcal{Q}_{\mu\nu}(x) = \left\langle x | \hat{\mathcal{Q}}'_{\mu\nu} | 0 \right\rangle \,. \tag{126}$$

Using these definitions, we obtain

$$\begin{aligned}\hat{\mathcal{Q}}'_{\mu\nu} &= \left\langle p_\mu p_\nu N \right\rangle_p + \frac{1}{6}\mathbf{R} \left\langle p_\mu p_\nu N^2 \right\rangle_p \\ &\quad + \left(Z^\alpha_{\ \nu} - \frac{1}{3}\mathcal{R}^\alpha_{\ \nu} \right) \left\langle p_\alpha p_\mu N^2 \right\rangle_p + R^{\alpha\ \ \beta}_{\ \mu\ \ \nu} \left\langle p_\alpha p_\beta N^2 \right\rangle_p \\ &\quad + \mathcal{O}\left(\nabla^4\right) \,.\end{aligned} \tag{127}$$

As shown in Appendix C the momentum integrals can be computed to yield formally the same result as in flat space-time, except that the flat metric is replaced by the metric tensor at x. It is often convenient to apply first angular averages, namely,

$$\left\langle p_\mu p_\nu f(p^2) \right\rangle_p = \left\langle -\frac{p^2}{d} g_{\mu\nu} f\left(p^2\right) \right\rangle_p \,; \tag{128}$$

or more generally

$$p_{\mu_1} \cdots p_{\mu_{2n}} \mapsto \frac{(-p^2)^n}{d(d+2)\cdots(d+2n-2)} g_{\mu_1 \cdots \mu_{2n}} \,, \tag{129}$$

where $g_{\mu_1 \cdots \mu_{2n}}$ is the completely symmetric sum of n-products of metrics ($(2n-1)!!$ terms).

The angular average yields

$$\begin{aligned}\hat{\mathcal{Q}}'_{\mu\nu} &= -\frac{1}{d} g_{\mu\nu} \left\langle p^2 N \right\rangle_p \\ &\quad - \frac{1}{d} \left(\frac{1}{6} g_{\mu\nu}\mathbf{R} + \frac{2}{3}\mathcal{R}_{\mu\nu} + Z_{\mu\nu} \right) \left\langle p^2 N^2 \right\rangle_p + \mathcal{O}\left(\nabla^4\right) \,.\end{aligned} \tag{130}$$

Finally, the standard formulas of dimensional integration apply,

$$\left\langle (p^2)^r N^n \right\rangle_p = \frac{(m^2)^{d/2+r-n}}{(4\pi)^{d/2}} \frac{\Gamma(d/2+r)}{\Gamma(d/2)} \frac{\Gamma(n-d/2-r)}{\Gamma(n)} \,, \tag{131}$$

and so

$$\begin{aligned}\hat{\mathcal{Q}}'_{\mu\nu} &= \frac{m^d}{(4\pi)^{d/2}} \Gamma(1-d/2) \\ &\quad \times \left[\frac{1}{d} g_{\mu\nu} - \frac{1}{m^2} \left(\frac{1}{12} g_{\mu\nu}\mathbf{R} + \frac{1}{3}\mathcal{R}_{\mu\nu} + \frac{1}{2} Z_{\mu\nu} \right) \right] \\ &\quad + \mathcal{O}\left(\nabla^4\right) \,.\end{aligned} \tag{132}$$

[19] Terms with four p cancel among them. It is not obvious to me whether this is just accidental or to be expected a priori.

This result is manifestly covariant, and formally independent of the domain and range of $\hat{\mathcal{Q}}_{\mu\nu}$, as $\hat{\mathcal{Q}}_{\mu\nu}$ itself in (113). To fully fix the matrix element it remains to specify the internal and world structures of the states $\langle x, a, w|$ and $|0, b, w'\rangle$.

7 Concluding remarks

From the computational point of view, our main result is contained in (90) and (66) (for a general connection), and (111) and (112) for the Riemannian connection. With such building blocks, and with the help of the representation (homomorphism) property, one can construct the covariant symbol of other operators $f(\nabla, M)$. This has been illustrated with an explicit computation in Sect. 6. It clearly would be interesting to extend the present results to higher orders and to include torsion more systematically. See e.g. [54, 55] for relations between torsion and chirality. For references motivating the study of quantum field theory in curved space-time with torsion see [56–58].

Regarding the concrete expressions obtained, we observe that they are rather natural, involving local covariant operators, similar to the heat kernel coefficients.[20] However, we warn that the presence of $Z_{\mu\nu}$ (or other operators in $\mathcal{C}(Z)$) is unusual as compared to other treatments. In those treatments [16], if one needs to apply, say, the heat kernel operator on a state with world indices (e.g., the gluon field, G_μ), a first step is to transform the world index into an internal one using a tetrad field, $G_a = \mathrm{e}_a^\mu\, G_\mu$. The new field G_a is a coordinate scalar so one can apply the heat kernel expressions for scalars. The world structure of the field is now in the internal sector through the corresponding connection for the tetrad index. In this way the result depends on an $F_{\mu\nu}(x)$ which includes the strength tensor from the original gauge structure plus that of the new internal structure. The idea is to assimilate coordinate covariance as much as possible to the gauge case, where $F_{\mu\nu}(x)$ is a matrix-valued function. Our own representation is different since the expressions obtained in the present work hold regardless of the world tensor structure of the states, without transforming them into scalars. This works thanks to the action of $Z_{\mu_1\ldots\mu_n}$ which are not just matrix-valued fields: from the gauge point of view, while $F_{\mu\nu}$ is the same matrix-valued function in, e.g. $F_{\mu\nu}B_\alpha|\,\rangle$ and in $B_\alpha F_{\mu\nu}|\,\rangle$, $Z_{\mu\nu}$ would be a "different" matrix-valued function in each case, since it acts on any world index at its right. Of course, nothing prevents us to reduce our formulas to reproduce the abovementioned more usual point of view. To do so, once the state has been transformed into a world scalar, one only needs to move all $Z_{\mu\nu}$ to the right using commutators, and then set the world part of $Z_{\mu\nu}$ to zero (since it is acting on a world scalar). Nevertheless, in our view, it is more natural to work with the original fields rather than transforming them into scalars by means of an ad hoc new internal structure.

With respect to applications of the method exposed, it naturally applies to one-loop computations in curved space-time. A first application would be to compute the heat kernel, not using the standard Seeley–DeWitt expansion, which orders operators by their dimension, but the covariant derivative expansion. Explicit calculations along this line exist only for the minimal case (i.e., Klein–Gordon theories with a trivial gauge sector) [27, 53]. The non-minimal, but flat space-time, calculation of [19] can be extended to the curved case, and results for traced and untraced coefficients will be presented elsewhere. Further applications refer to the effective action of Klein–Gordon and Dirac theories in curved space-time. Again results obtained by the covariant symbol method exist for these two cases, for flat space-time and quite general non-Abelian backgrounds [48, 50]. These computations correctly reproduce the Wess–Zumino–Witten action [59, 60] as well as the associated anomalies in the abnormal parity sector of the fermion case. For Dirac fermions in curved space-time there are many interesting results concerning chiral, coordinate and frame anomalies [51, 61–64], but results are much more scarce for the effective action itself. We expect that all the anomalies will be obtained as a byproduct of the effective action computation.

Mathematically, the covariant symbol is a quite interesting and challenging quantity, since it implies the construction of a true representation of pseudodifferential operators in terms of purely multiplicative operators (in the original x space). It would be very nice to have any rigorous result concerning such quantities and in particular to obtain the exact covariant symbol in particular cases. The fact that the covariant symbol can be computed systematically within concrete (presumably asymptotic) expansions suggests that this quantity can be given a rigorous and proper mathematical definition.

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Appendix A: Notational conventions

In this appendix we summarize the non-standard notational conventions used in the main text.

Derivative convention. For a quantity X_I having an ordered set of world indices I, $X_{\mu I}$ denotes its covariant derivative $[\nabla_\mu, X_I]$. For example, $R_{\sigma\mu\nu}{}^\alpha{}_\beta := [\nabla_\sigma, R_{\mu\nu}{}^\alpha{}_\beta]$. Exceptions are $Z_{\mu_1\mu_2\cdots\mu_n}$, defined in (91) so that they are multiplicative operators with respect to x, and the tensors $P_{\mu_1\mu_2\cdots\mu_n}{}^\alpha{}_\beta$ defined in (64).

[20] It is noteworthy that actually not all such operators are present and thus selection rules are at work. For instance, terms of the form $M_{ss}\mathcal{R}_{ss}(\partial^s)^4$ do not appear in $\overline{M}$. Technically, the reason for the non-existence of such terms is that $\overline{M}$ does not involve p_μ (cf. (78)). Then, using only ∂^μ and derivatives of M, the Z, the Riemann tensor and the torsion, there is no way to contract all indices. The presence of p_μ would permit $[\partial^\nu, p_\mu] = \delta_\mu^\nu$, and hence the Ricci tensor to appear. This is an alternative proof of (90).

Note that on states (wavefunctions) the action of ∇_μ is expressed as $\nabla_\mu\psi$ while on operators it acts adjointly, $[\nabla_\mu, X]$. Occasionally will write simply $\nabla_\mu X$ if $X \in \mathcal{C}(\underline{\nabla}, \underline{Z}, \underline{I})$, e.g. $\nabla_\sigma R_{\mu\nu}{}^\alpha{}_\beta$ or $\nabla_\mu t^A_\nu$.

Momentum convention. The momentum p_μ is purely imaginary, $p_\mu = \mathrm{i}k_\mu$ (k_μ real). However, $\mathrm{d}^d p$ is just the standard real measure $\mathrm{d}^n k$ and $p^2 := -g^{\mu\nu}p_\mu p_\nu = g^{\mu\nu}k_\mu k_\nu$ the standard real norm (positive for Euclidean signature).

s index convention. s indicates a symmetrized world index. So we will use the notation

$$A_{s\mu ss}(\partial^s)^3 \tag{A.1}$$

to mean

$$A_{\alpha\mu\beta\gamma}\partial^\alpha\partial^\beta\partial^\gamma\,. \tag{A.2}$$

There is no ambiguity since the ∂^μ commute.[21] In Sect. 5 we use a similar convention for the index S, which refers to the labels of the type A, B, etc.

Riemann tensor convention. For a gauge singlet and world vector wavefunction

$$[\nabla_\mu, \nabla_\nu]V^\lambda = +R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma - T_{\mu\nu}{}^\sigma\,\nabla_\sigma V^\lambda\,; \tag{A.3}$$

the Ricci tensor and the scalar curvature are

$$\mathcal{R}_{\mu\nu} := R_{\lambda\mu}{}^\lambda{}_\nu\,, \quad \mathbf{R} := g^{\mu\nu}\mathcal{R}_{\mu\nu}\,. \tag{A.4}$$

Appendix B: Computation of $\mathcal{Q}_{\mu\nu}(x)$ using the method of symbols

In this appendix we will illustrate the method of (ordinary) symbols with the same operator $\hat{\mathcal{Q}}_{\mu\nu}$ considered in Sect. 6. (We will use definitions introduced in that section.) Equation (42) implies

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) &= \frac{1}{\sqrt{g^{(\xi)}(x)}}\int\frac{\mathrm{d}^d p_A}{(2\pi)^d}\langle x|\mathcal{Q}_{\mu\nu}(\nabla+p,M)|0\rangle\\ &= \Big\langle(\nabla_\mu+p_\mu)\frac{1}{m^2-(\nabla_\alpha+p_\alpha)(\nabla^\alpha+p^\alpha)}\\ &\quad\times(\nabla_\nu+p_\nu)\Big\rangle.\end{aligned} \tag{B.1}$$

Carrying out an expansion in powers of ∇_μ through second order yields

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle &p_\mu p_\nu N + p_\mu N\Big(\Delta + \big\{\nabla_\alpha, p^\alpha\big\}N\big\{\nabla_\beta, p^\beta\big\}\Big)Np_\nu\\ &+ p_\mu N\left\{\nabla_\alpha, p^\alpha\right\}N\nabla_\nu + \nabla_\mu N\left\{\nabla_\alpha, p^\alpha\right\}Np_\nu\\ &+ \nabla_\mu N\nabla_\nu\Big\rangle + \mathcal{O}\left(\nabla^4\right)\,.\end{aligned} \tag{B.2}$$

The next step is to move all ∇ to (say) the right (it is essential not to split the covariant derivative into non-covariant pieces). The move is obtained by applying the rules

$$[\nabla_\mu, p_{\alpha_1\cdots\alpha_n}] = p_{\mu\alpha_1\cdots\alpha_n}\,, \quad [\nabla_\mu, N] = 2p^\alpha p_{\mu\alpha}N^2\,, \tag{B.3}$$

this gives

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle &p_\mu p_\nu N + p_\mu p^\alpha_{\alpha\nu}N^2 + p_\nu p_\mu{}^\alpha{}_\alpha N^2 + 2p^\alpha p_{\mu\alpha\nu}N^2\\ &+ 2p_\mu p_\nu p^\alpha p_\alpha{}^\beta{}_\beta N^3 + 2p_\mu p_\nu p^\alpha p^\beta{}_{\beta\alpha}N^3\\ &+ 4p_\mu p^\alpha p^\beta p_{\alpha\beta\nu}N^3 + 4p_\nu p^\alpha p^\beta p_{\mu\alpha\beta}N^3\\ &+ 8p_\mu p_\nu p^\alpha p^\beta p^\gamma p_{\alpha\beta\gamma}N^4 + N\nabla_\mu\nabla_\nu\\ &+ p_\mu p_\nu N^2\nabla^\alpha\nabla_\alpha + 2p_\mu p^\alpha N^2\nabla_\alpha\nabla_\nu\\ &+ 2p_\nu p^\alpha N^2\nabla_\mu\nabla_\alpha + 4p_\mu p_\nu p^\alpha p^\beta N^3\nabla_\alpha\nabla_\beta\\ &+ \text{ 39 further terms}\Big\rangle + \mathcal{O}\left(\nabla^4\right)\,.\end{aligned} \tag{B.4}$$

The "39 further terms" not made explicit contain a factor $p_{\alpha\beta}$. For the Riemannian connection they vanish by choosing normal coordinates centered at x. (Of course, the rule $p_{\alpha\beta}=0$ can only be applied after all covariant derivatives have been put aside.)

The derivatives of p_μ are easily obtained recalling that $p_\alpha = t^A_\alpha p_A$; thus

$$p_{\mu_1\cdots\mu_n} = t^A_{\mu_1\cdots\mu_n}p_A = t^A_{\mu_1\cdots\mu_n}t^\lambda_A\,p_\lambda\,. \tag{B.5}$$

In particular, using (96) one obtains

$$p_{\alpha\mu\nu} = \frac{1}{3}\left(R^\lambda_{\mu\nu\alpha} + R^\lambda_{\nu\mu\alpha}\right)p_\lambda\,. \tag{B.6}$$

Substitution in the expression above gives

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle &p_\mu p_\nu N - \frac{2}{3}\mathcal{R}_{\mu\alpha}\,p_\nu p^\alpha N^2 + \frac{1}{3}\mathcal{R}_{\nu\alpha}\,p_\mu p^\alpha N^2\\ &+ \frac{2}{3}R_{\mu\alpha\nu\beta}\,p^\alpha p^\beta N^2 - \frac{2}{3}\mathcal{R}_{\alpha\beta}\,p_\mu p_\nu p^\alpha p^\beta N^3\\ &+ N\nabla_\mu\nabla_\nu + p_\mu p_\nu N^2\nabla^\alpha\nabla_\alpha + 2p_\mu p^\alpha N^2\nabla_\alpha\nabla_\nu\\ &+ 2p_\nu p^\alpha N^2\nabla_\mu\nabla_\alpha + 4p_\mu p_\nu p^\alpha p^\beta N^3\nabla_\alpha\nabla_\beta\Big\rangle\\ &+ \mathcal{O}\left(\nabla^4\right)\,.\end{aligned} \tag{B.7}$$

Prior to momentum integration, this expression is not manifestly covariant since there are still ∇ not derivating anything nor in the form of $Z_{\mu_1\cdots\mu_n}$, that is, the expression in brackets is not a multiplicative operator. It is often not necessary to completely carry out the momentum integration to achieve manifest gauge covariance [17, 65]. Taking

[21] Note that in an expression like $\partial^\alpha Z_{\alpha\beta\mu}\partial^\beta$ the indices α and β are not symmetrized (Z and ∂ do not commute). The expression differs from $\partial^\beta Z_{\alpha\beta\mu}\partial^\alpha$, and hence it would not be faithfully represented by $\partial^s Z_{ss\mu}\partial^s$.

an angular average, as explained in Sect. 6, gives

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle & -\frac{p^2}{d}g_{\mu\nu}N - \frac{1}{3}\frac{p^2}{d}\mathcal{R}_{\mu\nu}N^2 - \frac{2}{3}\frac{(p^2)^2}{d(d+2)}g_{\mu\nu}\mathbf{R}N^3 \\ & - \frac{4}{3}\frac{(p^2)^2}{d(d+2)}\mathcal{R}_{\mu\nu}N^3 + N\nabla_\mu\nabla_\nu \\ & - 4\frac{p^2}{d}N^2\nabla_\mu\nabla_\nu - \frac{p^2}{d}g_{\mu\nu}N^2\nabla^\alpha\nabla_\alpha \\ & + 4\frac{(p^2)^2}{d(d+2)}N^3\nabla_\mu\nabla_\nu + 4\frac{(p^2)^2}{d(d+2)}N^3\nabla_\nu\nabla_\mu \\ & + 4\frac{(p^2)^2}{d(d+2)}g_{\mu\nu}N^3\nabla^\alpha\nabla_\alpha\Big\rangle + \mathcal{O}\left(\nabla^4\right) . \end{aligned} \tag{B.8}$$

Using now the recurrence

$$\left\langle (p^2)^r N^n \right\rangle_p = \frac{d/2+r-1}{n-1}\left\langle (p^2)^{r-1}N^{n-1}\right\rangle_p , \qquad n>1 \tag{B.9}$$

to eliminate higher powers of N, gives already a covariant result:

$$\begin{aligned}\mathcal{Q}_{\mu\nu}(x) = \Big\langle & -\frac{p^2}{d}g_{\mu\nu}N - \frac{1}{12}g_{\mu\nu}\mathbf{R}N - \frac{1}{3}\mathcal{R}_{\mu\nu}N \\ & - \frac{1}{2}N[\nabla_\mu,\nabla_\nu]\Big\rangle + \mathcal{O}\left(\nabla^4\right) . \end{aligned} \tag{B.10}$$

Upon momentum integration this coincides with (132) obtained in Sect. 6 using the method of covariant symbols.

Appendix C: Momentum integrals

Here we want to show that momentum integrals like (125) give formally the same result as in flat space-time. Rather than considering the most general case, it will be sufficient to treat a sample integral. We consider

$$I_{\mu\nu} = \frac{1}{\sqrt{g}}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, k_\mu k_\nu f\left(k^2\right) . \tag{C.1}$$

Here k_μ are real ($p_\mu := \mathrm{i}k_\mu$, $\mathrm{d}^d p := \mathrm{d}^d k$; cf. Appendix A) and $k^2 = g^{\alpha\beta}k_\alpha k_\beta$.

Introducing an orthonormal tetrad, e^a_α, as well as a new momentum variable, q_a,

$$\mathrm{e}^\alpha_a\ \mathrm{e}^a_\beta = \delta^\alpha_\beta , \qquad g_{\alpha\beta} = \eta_{ab}\mathrm{e}^a_\alpha\mathrm{e}^b_\beta , \qquad k_\alpha = \mathrm{e}^a_\alpha q_a , \tag{C.2}$$

we obtain

$$\begin{aligned} & g = |\det g_{\alpha\beta}| = |\det \mathrm{e}^a_\alpha|^2 , \quad \mathrm{d}^d k = |\det \mathrm{e}^a_\mu|\,\mathrm{d}^d q = \sqrt{g}\,d^d q , \\ & k^2 = \eta^{ab}q_a\ q_b = q^2 , \end{aligned} \tag{C.3}$$

so

$$I_{\mu\nu} = I_{ab}\,\mathrm{e}^a_\mu\mathrm{e}^b_\nu , \tag{C.4}$$

with

$$I_{ab} = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, q_a\ q_b f(q^2) = \eta_{ab}I , \tag{C.5}$$

and finally

$$I_{\mu\nu} = g_{\mu\nu}I , \tag{C.6}$$

with I computed as in flat space-time

$$I = \int \frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{d}q^2 f(q^2) . \tag{C.7}$$

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